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On the Chebyshev spectral collocation method in channel and jet flows

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Abstract

This paper deals with the application of spectral methods in solving the Orr-Sommerfeld and Rayleigh equations in two dimensional flows. The Orr-Sommerfeld equation for plane Poiseulle flow is solved and the curves of marginal stability in the (α, Re) and (c, Re) - planes (α, the wavenumber, c , the wavespeed and Re, Reynolds number) are calculated using the Chebyshev Spectral Collocation (CSC) method; the eigen-solutions are compared with results from previous studies. Also, the inviscid instability characteristics of a two dimensional jet flow, the Bickley Jet flow, are studied using the CSC technique. Some discussion is given on the extent of applicability of the CSC in such stability/instability problems.

1 Introduction

The use of spectral methods in the solution of ordinary and partial differential equations has increased in recent years. The popularity of spectral methods comes from the fact that they have been proven to be more accurate than finite difference and finite element numerical schemes (see for example Orszag 1971 , Gotlieb and Orszag 1977, Canuto *et al.* 1988). The reason behind the efficiency of spectral method schemes is their promise of *spectral accuracy*; that is for problems with smooth solutions convergene rates of $O(e^{-cN})$ or $O(e^{-c\sqrt{N}})$ are routinely achieved, where N is the number of degrees of freedom in the expansions (usually trigonometric or polynomial global interpolants for spectral methods and local interpolants such as piece-wise polynomials in finite elements or finite differences) used to approximate the unknown solution of the differential equation. On the other hand, finite differences or finite elements yield convergence rates that are only algebraic in N, typically $O(N^{-2})$ or $O(N^{-4})$.

The work presented in this paper is an application of a spectral method, the Chebyshev Spectral Collocation (CSC) method, in the solution of the Orr-Sommerfeld (OS) equation

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for plane Poiseulle flow and the Rayleigh equation for the Bickley Jet flow. The results for the OS equation are compared with the accurate results results of Orszag (1971) who worked on the same problem using a similar spectral method, the Chebyshev-Tau method. In both the Tau and the CSC methods, the exact derivatives in the differential equation are replaced by derivatives of interpolating polynomials at the Chebyshev points in the domain. However, the two methods differ in the manner in which the derivatives are computed at the Chebyshev points. In the Chebyshev-Tau method transform-recursion algorithms are used in the computation of the derivatives whereas in the present method we use matrix-vector multiplication to compute derivatives at the Chebyshev points (hereinafter refered to as Collocation points). Recent years have seen a widespread use of the matrix-vector multiplication technique in the computation of Chebyshev derivatives of ordinary and partial differential equations (see for example, Khorami *et al.* 1989, Karageorghis and Phillips 1991, Don and Solomonoff 1995a, 1997b among others). The advent of software packages which are powerful in matrix operations, like MATLAB, has made is more convenient and easier to apply spectral ideas via matrices than transform methods, like Fast Fourier Transforms.

This paper is organised as follows: In Section 2 we present the governing equation for the model problem and the linear stability theory for the OS equation. In Section 3 we present the numerical schemes and algorithms which lead to the solution of the OS equation. In Section 4 we present the inviscid instability analysis of the Rayleigh equations for the Bickley Jet and we compare our results with the numerical results of Drazin and Howard (1966). Finally, in Section 5 we give a discussion of the results and the conclusion.

2 Mathematical model

2.1 Problem formulation

We consider the problem of an incompressible fluid flowing in a two dimensional channel. The equations governing the motion of the fluid are the following Navier-Stokes equations :

$$\begin{aligned} \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} &= 0 \quad , \\ \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} &= -\frac{\partial p}{\partial x}+\frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right), \\ \frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y} &= -\frac{\partial p}{\partial y}+\frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right). \end{aligned} \tag{1}$$

where x and y are the streamwise and normal coordinates respectively, u and v are the streawise and normal velocity components respectively, t is time, p is pressure and Re is the Reynolds number. The flow quantities in equation (1) have been non-dimensionalized as follows : all velocities have been non-dimensionalized by the characteristic speed U which can be taken to be the centreline speed, the distances by a charateristic length L which is half the channel width.

The channel walls are located at $y = \pm 1$ and the basic flow in the x direction is given by $U(y) = 1 - y^2$. The boundary conditions (1) are the no-slip conditions, that is, the flow

* quantities are zero at the walls.

2.2 Stability Theory and Analysis

We introduce small disturbances to the basic flow quantities as follow:

$$u = U + u', \qquad v = v', \qquad p = P + p', \tag{2}$$

where u', v', p' are very small ($<< 1$).

Substituting (2) in (1) and using linearizing (neglecting quadratic terms in the disturbances) yields;

$$\begin{aligned}
\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} &= 0 \ , \\
\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + v'\frac{dU}{dy} &= -\frac{\partial p'}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 u'}{\partial y^2}\right), \\
\frac{\partial v'}{\partial t} + U\frac{\partial v'}{\partial x} &= -\frac{\partial p'}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v'}{\partial x^2} + \frac{\partial^2 v'}{\partial y^2}\right).
\end{aligned} \tag{3}$$

We assume that the disturbances are periodic and we define the stream function for two dimensional flow as

$$\psi(x, y, t) = \phi(y)e^{i\alpha(x-ct)} \tag{4}$$

where α is the wavenumber and c is the wave speed.

The velocity components can be expressed in terms of the stream function as follows,

$$u' = \frac{\partial \psi}{\partial y} = \phi'(y)e^{i\alpha(x-ct)}, \tag{5}$$

$$v' = -\frac{\partial \psi}{\partial x} = -i\alpha\phi(y)e^{i\alpha(x-ct)} \tag{6}$$

where the prime denotes differentiation with respect to y. Substituting (5) and (6) in (3) and eliminating pressure terms yields

$$(U - c)(\phi'' - \alpha^2\phi) = U''\phi = -\frac{i}{\alpha Re}\left(\phi^{(iv)} - 2\alpha^2\phi'' + \alpha^4\phi\right). \tag{7}$$

This is the Orr-Sommerfeld equation. The boundary condition of (7) are,

$$\phi(-1) = \phi'(-1) = 0, \tag{8}$$

$$\phi(1) = \phi'(1) = 0. \tag{9}$$

It will not be possible to find a non-trivial function ϕ satisfying (7) and the boundary conditions (8) and (9) unless the parameters α, Re and c satisfy a certain complex eigenvalue relation, say

$$F(\alpha, c, Re) = 0. \tag{10}$$

If it is assumed that the disturbances are temporally growing, then α is real and c is complex. For most physically reasonable velocity profiles (10) can be solved for c as a function of the real parameters α and Re :

$$c = c_r(\alpha, Re) + ic_i(\alpha, Re). \tag{11}$$

The imaginary part of (11) determines whether the disturbances will grow or decay. When $\alpha c_i < 0$ the disturbances decay and when $\alpha c_i > 0$ the disturbances grow. If $\alpha c_i = 0$ the disturbances neither grow nor decay. In this case the disturbance modes are said to be neutrally stable. If, for a given basic velocity profile, $U(y)$, one or more neutrally stable modes (for which $c_i(\alpha, Re) = 0$) are present then the (α, Re) plane may be divided into regions where $c_i < 0$ and regions where $c_i > 0$. These regions are separated by the curve given by $c_i(\alpha, Re) = 0$, which is known as the *neutral (or marginal) stability curve.* To establish that these neutral curves represent the stability boundaries, it is necessary to show that c_i changes sign when crossing the neutral curve. Spartially growing modes (that is when c is real and α is complex) also exist for some velocity profiles but they are less common than the temporally growing modes.

In this work we shall use numerical techniques to find the neutral stability modes for two types of flows - Poiseulle and Jet flow.

3 Numerical Solution

To solve equation (7) subeject to the boundary conditions (8) and (9) we use the a spectral collocation method of Chebyshev type. To this end, we look for an approximate solution, ϕ^N, which is a global Chebyshev polynomial of degree N defined on the interval $[-1, 1]$ by

$$T_N(y) = \cos N\theta, \qquad \theta = \arccos y. \tag{12}$$

We use the popular Gauss-Lobatto collocation points to define the Chebyshev nodes in $[-1, 1]$, namely;

$$y_j = \cos\frac{\pi j}{N}, \qquad j = 0, 1, \ldots, N. \tag{13}$$

The function $\phi(y)$ is approximated by an interpolating polynomial which is constructed in terms of the values of ϕ at each of the collocation points by employing a truncated Chebyshev series of the form.

$$\phi(y) = \phi_N(y_j) = \sum_{k=0}^{N} \tilde{\phi}_k T_k(y_j), \qquad j = 0, 1, \ldots, N \tag{14}$$

where $\tilde{\phi}_k$ represents the series coefficients. Derivatives of the functions ϕ_N at the collocation points may be represented by

$$\frac{d\phi_N}{dy}(y_j) = \sum_{k=0}^{N} \mathbf{D}_{jk}\tilde{\phi}_k \tag{15}$$

where the derivative matrix $\mathbf{D}$ is given by

$$\mathbf{D}_{jk} = \frac{c_j}{c_k}\frac{(-1)^{j+k}}{y_j - y_k} \qquad j \neq k; j, k = 0, 1, \ldots, N,$$

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$$\mathbf{D}_{kk} = -\frac{y_k}{2(1-y_k^2)} \qquad k = 1, 2, \ldots, N-1,$$
$$\mathbf{D}_{00} = \frac{2N^2+1}{6} = -D_{NN},$$

where c_j and c_k are equal to 1 for $j, k = 1, 2, \ldots, N-1$ and $c_0 = c_N = 2$ (Canuto *et al.* 1988). Higher order derivatives are computed as simply multiple powers of $\mathbf{D}$, that is

$$\frac{d^i\phi_N}{dy^i}(y_j) = \sum_{k=0}^{N} \mathbf{D}^i{}_{jk}\tilde{\phi}_k, \quad k = 0, 1, \ldots, N \tag{16}$$

where i is the order of the derivative.

Substituting equations (12 - 16) in equations (7 - 9) yields

$$(\mathbf{A} - c\mathbf{B})\Phi = 0 \tag{17}$$

where

$$\mathbf{A} = \frac{i}{\alpha Re}(\mathbf{D}^4 - 2\alpha^2\mathbf{D}^2 + \alpha^4\mathbf{I}) + \mathbf{U}(\mathbf{D}^2 - \alpha^2\mathbf{I}) - \mathbf{U}'',$$
$$\mathbf{B} = \mathbf{D}^2 - \alpha^2\mathbf{I}.$$

Also, $\mathbf{I}$ is an $(N+1)\times(N+1)$ identity matrix, $\mathbf{U}$ = diag $U(y_j)$, $\mathbf{U}''$ = diag $U''(y_j)$ and , $\Phi = [\tilde{\phi}_0, \tilde{\phi}_1, \ldots, \tilde{\phi}_N]^T$. Here, T is transpose and diag( ) means that the entries of ( ) are placed on the main diagonal of an $(N+1)\times(N+1)$ matrix with the rest of the entries of the matrix being zero.

The boundary condtions become,

$$\tilde{\phi}_0 = 0, \quad \sum_{k=0}^{N} \mathbf{D}_{0k}\tilde{\phi}_k = 0 \quad \text{on } y = 1 \tag{18}$$
$$\tilde{\phi}_N = 0, \quad \sum_{k=0}^{N} \mathbf{D}_{Nk}\tilde{\phi}_k = 0 \quad \text{on } y = -1 \tag{19}$$

The boundary conditions (18) and (19) are imposed on (17) by setting

$$\hat{\mathbf{D}}\Phi = \begin{cases} \mathbf{D}\Phi & j = 1, \ldots, N-1 \\ 0 & j = 0, N \end{cases} \tag{20}$$

where the $\mathbf{D}$ is the usual differential operator defined by (15). The differential equation is then imposed on $j = 1, \ldots, N-1$ and $\Phi = 0$ is imposed at $j = 0, N$. This solution algorithm leads to the generalized eigenvalue problem

$$(\hat{\mathbf{A}} - c\hat{\mathbf{B}})\Phi = 0 \tag{21}$$

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where

$$\hat{\mathbf{A}} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ \mathbf{D}_{0,0} & \mathbf{D}_{0,1} & \dots & \mathbf{D}_{0,N-1} & \mathbf{D}_{0,N} \\ -- & --- & --- & --- & -- \\ & & \mathrm{A} & & \\ -- & --- & --- & --- & -- \\ \mathbf{D}_{N,0} & \mathbf{D}_{N,1} & \dots & \mathbf{D}_{N,N-1} & \mathbf{D}_{N,N} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

$$\hat{\mathbf{B}} = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ -- & --- & --- & --- & -- \\ & & \mathrm{B} & & \\ -- & --- & --- & --- & -- \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \qquad \Phi = \begin{pmatrix} \phi_0 \\ \phi_1 \\ -- \\ \vdots \\ -- \\ \phi_{N-1} \\ \phi_N \end{pmatrix}.$$

The generalised eigenvalue equation (21) was solved using the MATLAB command function **eig**. The accuracy of the present matrix collocation technique was compared with that of the Chebyshev-Tau method of Orszag (1971) for the most unstable mode of the Orr-Sommerfeld equation in Plane Poiseulle for $\alpha = 1$, $Re = 10000$. The exact eigenvalue was found by Orszag (1971) and equals

$$0.23752649 + 0.00373967i$$

to eight decimal places. The numerical results obtained in this work are given in the table below :

Table 1. Eigenvalues of the most unstable mode of Plane Poiseulle flow for $\alpha = 1$, $Re = 10000$

$N+1$	c	computation time (sec)
30	0.23739952 + 0.00375098i	0.22
40	0.23751460 + 0.00374111i	0.44
50	0.23752612 + 0.00373920i	0.66
60	0.23752650 + 0.00373967i	0.98
62	0.23752649 + 0.00373967i	1.26
64	0.23752649 + 0.00373967i	1.43

From the table above it can be seen that the accuracy to within one part in 10^8 was achieved using $N = 62$ in the present method. Using $N = 40$ collocation points we then computed the neutrally stable modes. It was found that the critical values at which unstable eigenmodes begin to exist are

$$Re_c = 5770, \alpha_c = 1.021, c = 0.2640.$$

This results are in good agreement (to within one part in 10^3) with the more accurate results given in Orszag (1971). The accuracy of the results may be improved by using a

* bigger N value. The curves of neutral stability for the Plane Poiseulle flow are shown in Figures 1 and 2. Figures 1 and 2 show the variation of the Reynolds number against the wavespeed and wavenumber respectively.

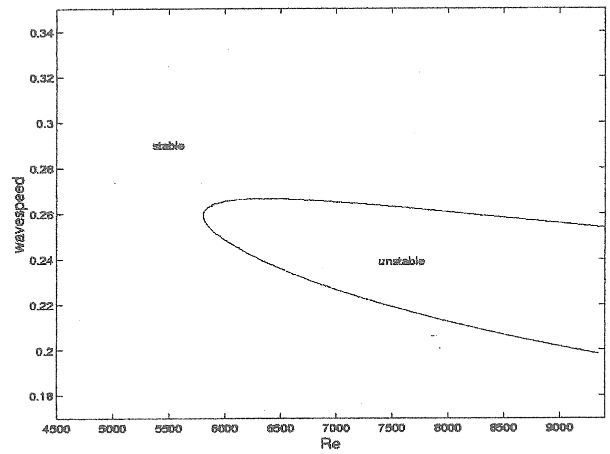

Figure 1: The relationship between the wave speed and Reynolds number along the curve of marginal stability for Plane Poiseulle Flow

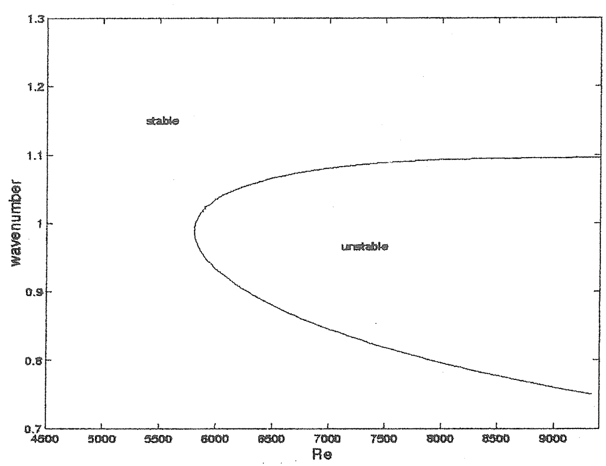

Figure 2: The relationship between the wavenumber and Reynolds number along the curve of marginal stability for Plane Poiseulle Flow

4 Inviscid Instability

In this section we discuss the instability of an inviscid "jet flow" governed by the Rayleigh equation

$$(U-c)(\phi''-\alpha^2\phi)-U''\phi=0. \tag{22}$$

which is the result of setting $\frac{1}{\alpha Re}=0$ in equation (7).

The flow under consideration is the Bickley Jet with the following basic velocity profile

$$U(y)=\text{sech}^2 y \qquad -\infty<y<\infty. \tag{23}$$

The boundary conditions are

$$\phi(-\infty)=\phi(\infty)=0. \tag{24}$$

To solve equation (22) subject to the boundary conditions (24) we follow the approach which was used to solve (7). However, here we focus on the inviscid instability characteristics of (23). In particular, we study the characteristics of the most unstable mode (the mode with the largest value of c_i). The symmetry of the basic flow makes it possible to consider even and odd solutions of (22) seperately. If we consider even and odd solutions separately then we restrict our attention to the interval $0\le z<\infty$. The even solution must satisfy the boundary conditions

$$\phi'(0)=0 \qquad \phi(\infty)=0 \tag{25}$$

and the odd solution must satisfy the boundary conditions

$$\phi(0)=0 \qquad \phi(\infty)=0. \tag{26}$$

The asymptotic behaviour $\phi(y)\sim e^{-\alpha y}(y\to\infty)$ may be used to infer an improved boundary at $y=\infty$

$$\phi'(\infty)+\alpha\phi(\infty)=0. \tag{27}$$

To transform the semi-infinite domain $[0,\infty]$ to the finite Chebyshev domain $[-1,1]$ we use the algebraic mapping

$$y=\frac{L(1+z)}{1-z} \tag{28}$$

where $L>0$ defines the scale factor of the mapping. Using this algebraic transformation, the Rayleigh equation becomes;

$$(U-c)\left(\xi^2\frac{d^2\phi(z)}{dz^2}+\xi'\frac{d\phi(z)}{dz}-\alpha^2\phi(z)\right)-U''\phi(z)=0, \tag{29}$$

where $\xi=\frac{dz}{dy}$ and $\xi'=\frac{d^2z}{dy^2}$. The boundary condition become

$$\frac{d\phi(-1)}{dz}=0, \quad \xi\frac{d\phi(1)}{dz}+\alpha\phi(1)=0, \tag{30}$$

for the even solution, and

$$\phi(-1)=0, \quad \xi\frac{d\phi(1)}{dz}+\alpha\phi(1)=0. \tag{31}$$

Applying the Chebyshev Collocation Spectral approximation procedure, as was applied in
∗ the previous section, on equations (29 - 34) we obtain the generalized eigenvalue problem

$$(\mathbf{E} - c\mathbf{F})\mathbf{\Phi} = 0 \tag{32}$$

where

$$\begin{aligned}\mathbf{E} &= \mathbf{U}(\xi^2\mathbf{D}^2 + \xi'\mathbf{D} - \alpha^2\mathbf{I}) - \mathbf{U}'',\\ \mathbf{E} &= \xi^2\mathbf{D}^2 + \xi'\mathbf{D} - \alpha^2\mathbf{I}.\end{aligned}$$

Note that $\xi = \text{diag}(\xi)$, $\xi' = \text{diag}(\xi')$ and the other matrices are as defined in the previous section.

The boundary conditions for the even and odd solutions become

$$\sum_{k=0}^{N} \mathbf{D}_{Nk}\phi_k = 0, \qquad \xi\sum_{k=0}^{N} \mathbf{D}_{0k}\phi_k + \alpha\phi_0 = 0 \tag{33}$$

and

$$\phi_N = 0, \qquad \xi\sum_{k=0}^{N} \mathbf{D}_{0k}\phi_k + \alpha\phi_0 = 0 \tag{34}$$

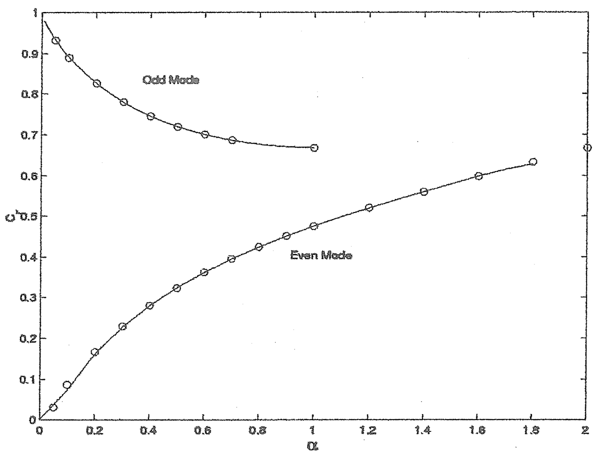

Figure 3: Plot of the wavespeed of the even and odd modes as a function of wavenumber for the Inviscid Bickley Jet flow. The results of Drazin and Howard (1966) are indicated by the small circles.

Plots of the solutions of equations (32 - 34) are given in Figures 3 and 4. Figures and 4 show the plots of the real part of the wavespeed, c_r and the growth rate, αc_i, versus the wavenumber, α for both the even and odd modes. In both Figures the results from the present method are compared to the numerical results of Drazin and Howard (1966), which are also given in Drazin and Reid (1981, p234). The present results were obtained using $N = 40$ collocation points and $L = 0.3$.

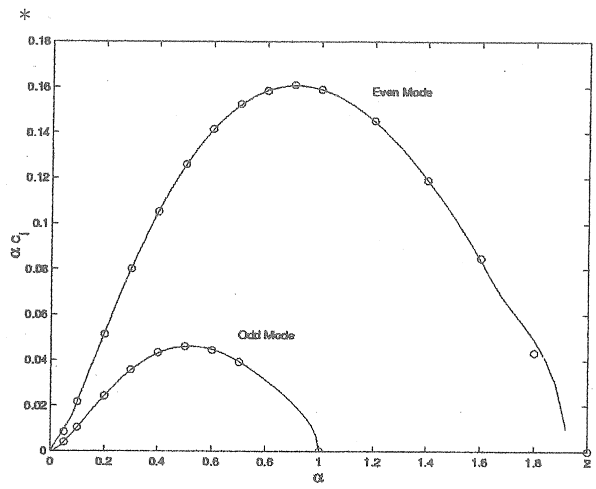

Figure 4: Plot of the growth rate of the even and odd modes as a function of the wavenumber for the Inviscid Bickley Jet flow. The results of Drazin and Howard (1966) are indicated by the small circles.

5 Discussion

In the present article we have applied the Chebyshev Spectral Collocation method to solve the Orr-Sommerfeld equation for Plane Poiseulle flow and the Rayleigh equation for the Bickley Jet flow. The results for the OS equation were compared to the results of Orszag (1971) who worked on the same problem using the Tau method. It was found that the method of using matrix-vector multiplication to compute Chebyshev derivatives gives accurate results provided that the number of collocation points used is taken to be sufficiently large. It should be noted however that the accuracy of the Tau method is better than that of the Collocation method employed in this paper. This is because the Collocation method is susceptible to round-off errors which arise in the process of computing higher order collocation derivatives using the matrix-vector multiplication procedure.

Breuer and Everson (1992) have pointed out that the most dangerous portion of errors associated with the computation of Collocation derivatives is located at the boundaries. They noticed that since the entries of the derivative matrix near the upper left and lower right corners are large they exert the greatest influence in the process of computing higher order derivatives. They proposed a preconditioning scheme which modified the process to reduce their influence and hence reduce the round-off errors. Another way of reducing round-off errors is through the use of grid transformation (see Don and Solomonoff 1997).

The results for the Rayleigh equation were compared to results obtained by Drazin and Howard (1966). From Figures 3 and 4 it can be seen that results for the even mode become less accurate when $\alpha \to 0$ and $\alpha \to \alpha_s$ (where α_s is the wavenumber of the mode associated with the inflexion point; $\alpha_s = 1$ for the odd mode and $\alpha_s = 2$ for the even mode). One difficulty which was encountered in the implementation of the CSC algorithm in the unbounded domain problem was choosing the best value of the mapping parameter, L. When certain values of L were used we failed to obtain some eigen-modes for some values *of α. Note that for the particular value of L used in this paper, eigen-modes are missing in the even solutions when $\alpha \to \alpha_s$.

We may conclude that the computer implementation of the CSC method on problems posed in unbounded domains is not as straightforward as that of problems posed on finite domains. The accuracy and convergence of the CSC method for the problems in unbounded domains may be improved by first using numerical search procedures (like the method of steepest descent, see Boyd 1982) to determine the optimal domain-size mapping parameter, L. The use of optimum mapping parameters in solving eigenvalue problems in unbounded domains is the subject of an ongoing study done by the authors. Also, it is the intention of the authors to investigate the use of preconditioning and grid transformation techniques to improve the accuracy of the collocation method employed in this paper.

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