Analysis of The Discrete Growth-Fragmentation Equation

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Abstract

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ANALYSIS OF THE DISCRETE GROWTH-FRAGMENTATION EQUATION

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Abstract: The discrete case of the growth-fragmentation equation is not as popular in the literature as the continuous form of it, hence the need to explore the discrete version of this model for some interesting analysis. In this paper, We examine the discrete growth-fragmentation equations. The problem is treated as an infinite-dimensional differential equation, posed in a suitable Banach space. Perturbation results, from the theory of strongly continuous semigroups of operators is used for the mathematical analysis of the model. In contrast to the truncation-limit approach, the semigroups of operator approach involves the use of operators to represent the growth-fragmentation model in order to establish the existence and uniqueness of solutions to the model.

Keywords: Fragmentation; Discrete; Growth; Model; Semigroups of Operators.

MSC: Mathematical physics, Mathematical biology, Applied Mathematics.

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1. Introduction

Fragmentation process consists of collection of aggregates of particles which can break into two (binary fragmentation) or more smaller clusters (multiple fragmentation). Each of these clusters is composed of a number of identical fundamental units called monomers. Many important world phenomena involve fragmentation processes, see [1–5]. When only fragmentation process is considered, it is often called pure fragmentation. A concrete example of a process that can occur with the fragmentation is growth. These two processes - growth and fragmentation, have two opposite dynamics. While growth leads the population towards larger sizes, fragmentation leads to smaller sizes. Similar to the case of pure fragmentation, we could also have a case when the growth process dominates. That is there is no fragmentation occurring. This is referred to as pure growth, where the population size grows naturally.

In the growth-fragmentation models the corresponding equations are designed to maintain a balance between growth and fragmentation, so as to keep the population around a finite size. The McKendrick-von Foerster renewal condition incorporated into the fragmentation equation is an important device that has many applications in natural sciences. The resulting models are used to describe evolution of a population in which individuals can grow, split or divide and die. There are two classes of McKendrick-von Foerster-type growth-fragmentation models:

- fragmentation equation with growth and mortality; and
- fragmentation equation with decay and mortality.

The discrete growth-fragmentation equation is given by:

$$\begin{aligned} \frac{du_i}{dt} &= r_{i-1}u_{i-1} - r_iu_i - d_iu_i - a_iu_i + \sum_{j=i+1}^{\infty} a_jb_{i,j}u_j, \quad i \geq 1, \\ u_i(0) &= u_i^0, \quad i \geq 1, \end{aligned} \tag{1}$$

where u_i is the number of clusters of mass i, $i \geq 1$, r_i and d_i represent the growth and mortality coefficients, $d_i \geq 0$, and $r_i > 0$. The fragmentation rate is given by a_i and $b_{i,j}$ is the average number of i-mers produced after the breakup of a j-mer, with $j \geq i$. Further, as clusters can fragment into two or more smaller pieces (multiple fragmentation) but not into bigger clusters, we note the following:

$$\begin{aligned} &a_1 = 0, \quad a_i \geq 0 \in \mathbb{R}, \quad i \geq 2, \\ &b_{1,j} = 0, \\ &b_{i,j} = 0, \quad i \geq j. \end{aligned} \tag{2}$$

The birth term is given by $r_0u_0(t) = \sum_{j=1}^{\infty} \beta_ju_j(t)$, where β_j is the rate at which cell of size j produces monomers.

The discrete decay-fragmentation equation is given by:

$$\begin{aligned} \frac{du_i}{dt} &= r_{i+1}u_{i+1} - r_iu_i - d_iu_i - a_iu_i + \sum_{j=i+1}^{\infty} a_jb_{i,j}u_j, \quad i \geq 1, \\ u_i(0) &= u_i^0, \quad i \geq 1, \end{aligned} \tag{3}$$

where r_i and d_i represent the decay and mortality coefficients.

Note that the total mass in the system at time t is given by $M(t) = \sum_{i=1}^{\infty} iu_i$ and for mass to be conserved we require that

$$\sum_{i=1}^{j-1} ib_{i,j} = j, \quad j \geq 2. \tag{4}$$

The fragmentation processes in (1) and (3) are mass conservative, but the growth/decay and mortality parts are not. Hence, there is a mass leakage or increase in the system. This study focuses only on fragmentation equation with growth and mortality, which is simply called growth-fragmentation equation.

2. Review of Literature

Most literature [5–13] on the growth-fragmentation equation deals with the continuous version of the model. Hence, the review will be for the continuous form of equation (1). The growth-fragmentation equation is also referred to as the transport-fragmentation equation [5,7].

Banasiak, [6], carried out the analysis of the continuous model in the space

$$X_{0,1} = L_1([x_0, x_1), (1+x)dx) = L_1([x_0, x_1), dx) \cap L_1([x_0, x_1), xdx),$$

where $0 \leq x_0 < x_1 \leq \infty$, where x_0 and x_1 are the minimum and maximum size of particles. The existence and uniqueness of a strongly differentiable solution was proved in this space using the theory of semigroups of operators. The same author, in [14], analyzed the inter-relation between the growth and fragmentation term of the phytoplankton aggregates disregarding the coagulation part. It was shown that the growth-fragmentation model displayed the typical properties of fragmentation models, especially when the fragmentation rate is unbounded as the size of aggregates tends to zero.

In [8], the continuous form of (1) was considered which represents the linear part of the equation decribing the dynamical behaviour of phytoplankton cells. Using the theory of semigroups of linear operators, the author examined the wellposedness of the phytoplankton equation in the space

$$X_1 = L_1\Big([0,\infty), xdx\Big) = \Big\{u; \int_0^\infty |u(x)|xdx < +\infty\Big\}.$$

Mischler and Scher [11], provided a spectral analysis of semigroups in a Banach space for three classes of the growth-fragmentation equation namely: the cell division equation with constant growth, the self-similar fragmentation equation and the McKendrick-Von Foerster age structured population equation. In [9], the author considered the linear growth-fragmentation equation for constant coefficient. Its convergence to a steady state with an exponential rate was proved via the weak compactness approach in L^1 settings.

Thibault et al. [12], in their study of the growth-fragmentation equation, considered the problem of finding the division rate of the growth-fragmentation equation in the case where the fragmentation depends only on the ratio between the parent size and the offspring size. The inverse problem method used in their study is based on the Mellin transform of the growth-fragmentation equation, in weighted L^2 spaces. Numerical results were provided to illustrate the theory.

We note that different methods of proof have been used to study the growth-fragmentation equation in the literature. Some of these methods are: Direct Laplace transform method [15,16], Weak Compactness and General Relative Entropy (GRE) method [17,18], Entropy-Dissipation of Entropy (E-DE) method [19,20], etc.

3. Semigroups of Operators

The tools needed to study the existence of a semigroup solution to the discrete growth-fragmentation equation are explained in this section. Semigroups of operators can be used to solve a large class of evolution equations which appear in many disciplines such as biology, physics, chemistry, and engineering. Hence, we introduce some basic definitions and results from the theory of semigroups which will be used in the treatment of the growth-fragmentation equation.

3.1. Semigroups and Generators of Semigroups

Definition 1. *A one-parameter family $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ of bounded linear operators on a Banach space X with norm $\|.\|$ is said to be a semigroup on X, if it satisfies*

1. $S(0) = I$,
2. $S(s)S(t) = S(s+t)$ *for all* $s, t \geq 0$.

Definition 2. *A semigroup $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is said to be uniformly continuous with respect to the operator norm $\|.\|$ associated with X, if*

$$\lim_{t\to 0^+} \|S(t) - I\| = 0,$$

while it is said to be strongly continuous or C_0-semigroup, if

$$\lim_{t\to 0^+} \|S(t)x - x\| = 0,$$

for all $x \in X$.

In view of Definition 1, we expect $S(t)$ to mimic the behaviour of classical exponential function. The following result shows that for C_0-semigroups this is indeed the case.

Theorem 1. *Let $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ be a C_0-semigroup, then there exist $M \geq 1$, $w \in \mathbb{R}$ such that*

$$\|S(t)\| \leq Me^{wt}, \quad t \geq 0. \tag{5}$$

The class of C_0-semigroups, that satisfies (5), is denoted by $C_0(M, w)$.

Proof. For the proof see [21,22]. □

Definition 3. *Let $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ be a C_0-semigroup. We say that $\{S(t)\}_{t\geq 0}$ is*

1. *a* C_0-semigroup of contractions *if $\{S(t)\}_{t\geq 0} \in C_0(1, 0)$;*
2. *a* positive semigroup *if $S(t)x \geq 0$ for all x in the positive cone X_+ of X;*
3. *a* sub-stochastic semigroup *if it is a semigroup of contractions and is positive;*
4. *a* stochastic semigroup *if it is positive and $\|S(t)x\| = \|x\|$, for all $x \in X_+$.*

Definition 4. *Let $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ be a C_0-semigroup. The linear operator A defined by the identity*

$$Ax = \lim_{t\to 0^+} \frac{S(t)x - x}{t}, \quad D(A) = \left\{x \in X : \lim_{t\to 0^+} \frac{S(t)x - x}{t} \text{ exists}\right\},$$

is called the infinitesimal generator of C_0-semigroup $\{S(t)\}_{t\geq 0}$.

Directly from Definition 4 it follows that to each C_0-semigroup can be assigned a unique infinitesimal generator $(A, D(A))$. The generator is linear but generally unbounded map from $D(A)$ to X. The converse is not true. Not every linear map $(A, D(A))$ is a generator of a C_0-semigroup. To be an infinitesimal generator of a C_0-semigroup, operator A must satisfy some extra conditions.

Theorem 2. *Let $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ be a C_0-semigroup, and let A be its generator. Then*

1. *the domain of A is dense in X;*
2. *the operator A is closed (or closable).*

Proof. See [23] for the proof. □

3.2. Generation Theorems

There is an intimate connection between the semigroup theory and Abstract Cauchy Problems (ACPs)

$$\frac{du}{dt} = Au, \quad u(0) = u_0,$$

posed in a Banach space X. In particular, if A is an infinitesimal generator of a C_0-semigroup $\{S(t)\}_{t\geq 0}$, then the strong solution of the corresponding ACP is given explicitly by $u(t) = S(t)u_0$, see []. Hence, arise the natural quaestion: under what conditions operator A generates a C_0-semigroup. Theorems that yield this are known as generation theorems. To formulate the results, we need the following

Definition 5. *The resolvent set $\rho(A)$ of the operator $(A, D(A))$ is given by*

$$\rho(A) = \{\lambda \in \mathrm{C} : R(\lambda; A) \in \mathcal{B}(X)\},$$

where $R(\lambda; A) = (\lambda I - A)^{-1}$ is called the resolvent operator associated with operator A.

Theorem 3 (Hille-Yosida)**.** *A closed linear operator $(A, D(A))$ on X with dense domain $D(A) \subseteq X$, generates a C_0-semigroup of contraction $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ if and only if $(0, \infty) \subseteq \rho(A)$ and the following resolvent inequality holds*

$$\|R(\lambda; A)\| \leq \frac{1}{\lambda}, \quad \lambda > 0.$$

Proof. See [21] for the proof. □

Theorem 4 (Feller-Miyadera-Phillips)**.** *A closed linear operator $(A, D(A))$ on X with dense domain $D(A) \subseteq X$, generates a C_0-semigroup $\{S(t)\}_{t\geq 0} \subset C_0(M, w)$ if and only if $(w, \infty) \subseteq \rho(A)$ and the following resolvent inequality holds*

$$\|R(\lambda; A)^n\| \leq \frac{M}{(\lambda - w)^n}, \quad \lambda > w, \quad n \geq 1.$$

Proof. For the proof, see [22,24]. □

3.3. Perturbation Theorems

Many practical problems take the form:

$$\frac{du}{dt} = Au + Bu, \quad u(0) = u_0.$$

In these settings, we view operator B as a perturbation of A. If A generates a C_0-semigroup and the perturbation B is not too 'bad' it is natural to expect that a suitable realization of the sum operator $A + B$ still generates a C_0-semigroup. Special literature contains large number of perturbation results. Below, we mention two that are relevant for our studies.

Theorem 5 (The Bounded Perturbation Theorem)**.** *Let $(A, D(A))$ be the infinitesimal generator of a $C_0(M, w)$-semigroup $\{S_A(t)\}_{t\geq 0}$ on X. If $B \in \mathcal{B}(X)$, then $(A + B, D(A))$ generates a $C_0(M, w + M\|B\|)$-semigroup $\{S_{A+B}(t)\}_{t\geq 0}$ on X. Moreover, the semigroup $\{S_{A+B}(t)\}_{t\geq 0}$ can be obtained via the Dyson-Philips expansion*

$$S_{A+B}(t)x = \sum_{n=0}^{\infty} V_n(t)x, \ \ V_0(t) = S_A(t), \ \ V_{n+1}(t)x = \int_0^t S_A(t-s)BV_n(s)xds, \ \ n \geq 0. \tag{6}$$

Proof. See [21,22]. □

The second perturbation result applies to positive C_0-semigroups.

Theorem 6. *A C_0-semigroup on X is positive if and only if the resolvent $R(\lambda; A)$ of its infinitesimal generator $(A, D(A))$ is positive for all sufficiently large λ.*

Proof. See [24]. □

Theorem 7 (Kato-Voigt Perturbation Theorem)**.** *Let $(A, D(A))$ and $(B, D(B))$ be two operators in $X = L^1(\Omega, \mu)$, where (Ω, μ) is a measure space. Assume further that*

1. *$(A, D(A))$ generates a sub-stochastic semigroup $\{S_A(t)\}_{t\geq 0}$;*
2. *$D(B) \supseteq D(A)$ and $Bx \geq 0$ for all $x \in D(B)_+ = D(B) \cap X_+$;*
3. *for all $x \in D(A)_+ = D(A) \cap X_+$*

$$\int_\Omega (Ax + Bx)d\mu \leq 0.$$

Then there exists a smallest sub-stochastic semigroup $\{S_G(t)\}_{t\geq 0}$ on X whose infinitesimal generator is an extension of $(A + B, D(A))$.

Proof. See [24]. □

We state a corollary of the Kato-Voigt Perturbation Theorem which will be useful for our analysis in this paper. Sometimes, the operator may generate a C_0-semigroup which is not sub-stochastic. Hence the need for the following corollary.

Corollary 1. *Let $(A, D(A))$ and $(B, D(B))$ be two operators in $X = L^1(\Omega, \mu)$, where (Ω, μ) is a measure space. Assume further that*

1. *$(A, D(A))$ generates a positive $C_0(1, w)$-semigroup $\{S_A(t)\}_{t\geq 0}$;*
2. *$D(B) \supseteq D(A)$ and $Bx \geq 0$ for all $x \in D(B)_+ = D(B) \cap X_+$;*
3. *for all $x \in D(A)_+ = D(A) \cap X_+$*

$$\int_\Omega (Ax + Bx) d\mu \leq w \int_\Omega x d\mu.$$

Then there exists a smallest positive $C_0(1, w)$-semigroup $\{S_G(t)\}_{t\geq 0}$ on X whose infinitesimal generator is an extension of $(A + B, D(A))$.

4. Analysis of the Discrete Linear Growth-Fragmentation Model

We shall apply the theory of semigroups of operators explained in the previous section to the growth-fragmentation equation (1). The associated ACP is given by

$$\frac{du}{dt} = Au + Bu, \quad u(0) = (f_n)_n^\infty. \tag{7}$$

In context of problem (1), the natural choice for the space is

$$X = \left\{ f := (f_n)_{n=1}^\infty \subset \mathbb{R} : \|f\| = \sum_{n=1}^\infty n|f_n| < \infty \right\}.$$

The norm $\|\cdot\|$ has simple physical interpretation — for a given system of particles $f = (f_n)_{n=1}^\infty$, $\|f\|$ is the total mass of the system. The operators $(A, D(A))$ and $(B, D(B))$ are defined as follow

$$\begin{aligned} [Af]_i &= r_{i-1} f_{i-1} - (r_i + d_i + a_i) f_i, \quad i \geq 1, \quad a_1 = 0, \\ [Bf]_1 &= \sum_{j=1}^\infty \beta_j f_j + \sum_{j=2}^\infty a_j b_{1,j} f_j, \\ [Bf]_i &= \sum_{j=i+1}^\infty a_j b_{i,j} f_j, \quad i \geq 2. \end{aligned}$$

In view of Corollary 1, the domains must be chosen so that $D(A) \subseteq D(B) \neq \emptyset$. We let

$$D(A) = \left\{ f \in X : r_1|f_1| + \sum_{n=2}^\infty n|r_{n-1} f_{n-1} - r_n f_n| < \infty, \quad \sum_{n=1}^\infty n(a_n + d_n)|f_n| < \infty \right\}, \tag{8a}$$

$$D(B) = \left\{ f \in X : \sum_{n=1}^\infty n a_n |f_n| < \infty \right\}. \tag{8b}$$

In general, the growth, the mortality and the fragmentation rates are variable. Below, we assume that the corresponding coefficients grow at most linearly, i.e.

$$0 \leq r_n \leq r.n, \quad 0 \leq d_n \leq d.n, \quad 0 \leq a_n \leq a.n, \tag{9}$$

for some fixed numbers $r, d, a > 0$. The wellposedness analysis proceeds in two steps: (i) we prove that the operator A does generate a C_0-semigroup; (ii) we apply the modified version of Kato-Voigt Perturbation Theorem to the sum $A + B$, to show that it also generate a C_0-semigroup.

4.1. The Growth Semigroup

To begin, we establish the existence of C_0-semigroup generated by the growth operator A. The ACP takes the form

$$\frac{du}{dt} = Au, \quad u(0) = (f_n)_n^\infty, \tag{10}$$

and represents a situation when no fragmentation occurs. The associated resolvent equation reads:

$$(\lambda + r_i + d_i + a_i)u_i - r_{i-1}u_{i-1} = f_i, \quad i \geq 1, \quad a_1 = 0. \tag{11}$$

The formal solution of (11) is given by

$$u_n = \sum_{i=1}^{n} \frac{f_i}{\lambda + r_i + d_i + a_i} \prod_{j=i}^{n-1} \frac{r_j}{\lambda + r_{j+1} + d_{j+1} + a_{j+1}}, \quad n \geq 1. \tag{12}$$

Using the operator notation, we write u_n in the form

$$[R_\lambda f]_n = \sum_{i=1}^{n} \frac{f_i}{\lambda + \theta_i} \prod_{j=i}^{n-1} \frac{r_j}{\lambda + \theta_{j+1}}, \quad \theta_n = r_n + d_n + a_n, \quad n \geq 1. \tag{13}$$

It turns out that the operator $R_\lambda : X \to X$, defined in (13), is bounded.

Lemma 1. *For $\lambda > \theta$, we have $\|R_\lambda\| \leq \frac{1}{\lambda - \theta}$, where $\theta = r + d + a$. Furthermore, $R_\lambda X \subset D(A)$.*

Proof. (a) First, we show that $R_\lambda : X \to X$ is bounded. Let $f \in X$ and $v = R_\lambda f$. We apply the triangle inequality and change the order of summation to obtain

$$\begin{aligned}
\|v\| &= \sum_{n=1}^{\infty} n \Big| \sum_{i=1}^{n} \frac{f_i}{\lambda + \theta_i} \prod_{j=i}^{n-1} \frac{r_j}{\lambda + \theta_{j+1}} \Big| \leq \sum_{i=1}^{\infty} |f_i| \sum_{n=i}^{\infty} \frac{n}{\lambda + \theta_n} \prod_{j=i}^{n-1} \frac{r_j}{\lambda + \theta_j} \\
&= \frac{1}{\lambda} \sum_{i=1}^{\infty} |f_i| \sum_{n=i}^{\infty} n \left(1 - \frac{\theta_n}{\lambda + \theta_n}\right) \prod_{j=i}^{n-1} \frac{r_j}{\lambda + \theta_j} \\
&= \frac{1}{\lambda} \sum_{i=1}^{\infty} |f_i| \left[\sum_{n=i}^{\infty} n \prod_{j=i}^{n-1} \frac{1}{\frac{\lambda}{\theta_j} + 1} - \sum_{n=i}^{\infty} n \prod_{j=i}^{n} \frac{1}{\frac{\lambda}{\theta_j} + 1} \right] \\
&= \frac{1}{\lambda} \sum_{i=1}^{\infty} |f_i| \left[i + \sum_{n=i+1}^{\infty} \prod_{j=i}^{n-1} \frac{1}{\frac{\lambda}{\theta_j} + 1} \right] \leq \frac{1}{\lambda} \sum_{i=1}^{\infty} |f_i| \left[i + \sum_{n=i+1}^{\infty} \left(\frac{j}{\frac{\lambda}{\theta} + j} \right)_{n-i} \right],
\end{aligned} \tag{14}$$

where $\theta = r + d + a$. The second sum in the last line of (14) can be computed explicitly. Indeed, with the aid of the identity $(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)}$, we infer

$$\begin{aligned}
\sum_{n=i+1}^{\infty} \left(\frac{j}{\frac{\lambda}{\theta} + j} \right)_{n-i} &= \sum_{n=i+1}^{\infty} \frac{\Gamma(\frac{\lambda}{\theta} + i)}{\Gamma(\frac{\lambda}{\theta} + n)} \frac{\Gamma(n-i)}{\Gamma(n-i)} = \frac{1}{(i-1)!} \sum_{n=i+1}^{\infty} \frac{(n-1)!}{(n-i-1)!} B\left(n-i, \frac{\lambda}{\theta} + i\right) \\
&= \frac{1}{(i-1)!} \int_0^1 (1-t)^{\frac{\lambda}{\theta}+i-1} \sum_{n=i+1}^{\infty} (n-i) \cdots (n-1) t^{(n-1)-i} dt \\
&= \frac{1}{(i-1)!} \int_0^1 (1-t)^{\frac{\lambda}{\theta}+i-1} \frac{d^i}{dt^i} \Big[\sum_{n=0}^{\infty} t^n \Big] dt = i \int_0^1 (1-t)^{\frac{\lambda}{\theta}-2} dt = \frac{i\theta}{\lambda - \theta}.
\end{aligned}$$

Upon the substitution into (14), the last formula yields the estimate

$$\|v\| \le \frac{\|f\|}{\lambda - \theta}.$$

(b) With an extra effort, it is not difficult to show that the range of R_λ is contained in $D(A)$, given in (8a). To simplify our calculations, we let $p_n = a_n + d_n$, $pf = (p_n f_n)_{n\ge 1}$ and $\triangle f = (f_1, (f_{n-1} - f(n))_{n\ge 2})$. First, we estimate $\|pv\|$. Our derivations are similar to those, used in formula (14):

$$\begin{aligned}
\|pv\| &= \sum_{n=1}^{\infty} n p_n \Big| \sum_{i=1}^{n} \frac{f_i}{\lambda + \theta_i} \prod_{j=i}^{n-1} \frac{r_j}{\lambda + \theta_{j+1}} \Big| \le \sum_{i=1}^{\infty} |f_i| \sum_{n=i}^{\infty} \frac{n(\theta_n - r_n)}{\lambda + \theta_n} \prod_{j=i}^{n-1} \frac{r_j}{\lambda + \theta_j} \\
&= \sum_{i=1}^{\infty} |f_i| \sum_{n=i}^{\infty} n \left[\frac{-\lambda}{\lambda + \theta_n} + 1 - \frac{r_n}{\lambda + \theta_n} \right] \prod_{j=i}^{n-1} \frac{r_j}{\lambda + \theta_j} \\
&\le \sum_{i=1}^{\infty} |f_i| \sum_{n=i}^{\infty} n \left[1 - \frac{r_n}{\lambda + \theta_n} \right] \prod_{j=i}^{n-1} \frac{r_j}{\lambda + \theta_j} \\
&= \sum_{i=1}^{\infty} |f_i| \left[i + \sum_{n=i+1}^{\infty} n \prod_{j=i}^{n-1} \frac{r_j}{\lambda + \theta_j} - \sum_{n=i}^{\infty} n \prod_{j=i}^{n} \frac{r_j}{\lambda + \theta_j} \right] \\
&= \sum_{i=1}^{\infty} |f_i| \left[i + \sum_{n=i+1}^{\infty} \prod_{j=i}^{n-1} \frac{r_j}{\lambda + \theta_j} \right] \le \frac{\lambda}{\lambda - \theta} \|f\|.
\end{aligned}$$

To complete the proof, we have to bound $\|\triangle rv\|$. For this, we observe that

$$\begin{aligned}
[\triangle rv]_n &= r_{n-1} v_{n-1} - r_n v_n = (\lambda + p_n) v_n + [r_{n-1} v_{n-1} - (\lambda + \theta_n) v_n] \\
&= (\lambda + p_n) v_n + \sum_{i=1}^{n-1} f_i \prod_{j=i}^{n-1} \frac{r_j}{\lambda + \theta_j} - \sum_{i=1}^{n} f_i \prod_{j=i}^{n-1} \frac{r_j}{\lambda + \theta_j} \\
&= (\lambda + p_n) v_n - f_n.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\|\triangle rv\| &= \|(\lambda + p)v - f\| = \|(\lambda + p_n) v_n - f_n\| \le \lambda \|v\| + \|pv\| + \|f\| \\
&\le \frac{\lambda}{\lambda - \theta} \|f\| + \frac{\lambda}{\lambda - \theta} \|f\| + \|f\| = \frac{3\lambda - \theta}{\lambda - \theta} \|f\|,
\end{aligned}$$

and we conclude that $R_\lambda X \subset D(A)$. □

In our next step, we show that R_λ is the resolvent of the operator $(A, D(A))$.

Lemma 2. *For $\lambda > \theta$, we have $R_\lambda = R(\lambda; A)$.*

Proof. (a) For $\lambda > \theta$, operator R_λ is the left inverse of $\lambda I - A$. Indeed, for any $u \in X$, we have

$$\begin{aligned}
[R_\lambda(\lambda I - A) u]_n &= \sum_{i=1}^{n} \frac{1}{\lambda + \theta_i} \prod_{j=i}^{n-1} \frac{r_j}{\lambda + \theta_{j+1}} ((\lambda + \theta_i) u_i - r_{i-1} u_{i-1}) \\
&= \sum_{i=1}^{n} u_i \prod_{j=i}^{n-1} \frac{r_j}{\lambda + \theta_{j+1}} - \sum_{i=2}^{n} u_{i-1} \prod_{j=i-1}^{n-1} \frac{r_j}{\lambda + \theta_{j+1}} = u_n.
\end{aligned}$$

(b) For $u \in X$, we let $v = R_\lambda u$. Thanks to Lemma 1, $v \in D(A)$, therefore $(\lambda I - A)v$ is well defined and we have

$$\begin{aligned}[(\lambda I - A)v]_n &= (\lambda + \theta_n)v_n - r_{n-1}v_{n-1} \\ &= \sum_{i=1}^{n} u_i \prod_{j=i}^{n-1} \frac{r_j}{\lambda + \theta_j} - \sum_{i=1}^{n-1} u_i \prod_{j=i}^{n-1} \frac{r_j}{\lambda + \theta_j} = u_n.\end{aligned}$$

Hence, for $\lambda > \theta$, R_λ is also the right inverse of $\lambda I - A$. We conclude that $(\theta, \infty) \subset \rho(A)$ and $R_\lambda = R(\lambda; A)$. □

Lemmas 1, 2 combined together yield the following result

Theorem 8. *The operator (A, X) generates a positive $C_0(1, \theta)$-semigroup $\{S_A(t)\}_{t\geq 0}$ on X.*

4.2. The Complete Model

Now we turn to the complete model (7). Since A generates $C_0(1, \theta)$-semigroup in the discrete L^1-type space, we can apply Corollary 1. Indeed, in view of (8), $D(B) \supseteq D(A)$. Since the coefficient of the model are nonnegative, it is also clear that $Bf \geq 0$ for any $f \in D(B)_+$. It remains to verify the last condition of the modified Kato-Voigt theorem. For $f \in D(B)_+$, we have

$$\begin{aligned}\sum_{n=1}^{\infty} n[Af + Bf]_n &= \sum_{n=1}^{\infty} n\left(r_{n-1}f_{n-1} - (r_n + d_n + a_n)f_n + \sum_{j=n+1}^{\infty} a_j b_{n,j} f_j\right) \\ &= r_0 f_0 + \sum_{n=2}^{\infty} n r_{n-1} f_{n-1} - \sum_{n=1}^{\infty} n r_n f_n - \sum_{n=1}^{\infty} n(a_n + d_n) f_n + \sum_{n=2}^{\infty} n a_n f_n \\ &= (\beta_1 + r_1 - a_1 - d_1) f_1 + \sum_{n=2}^{\infty} (\beta_n + r_n - n d_n) f_n \leq \max\{\theta, \beta + r\} \sum_{n=1}^{\infty} n f_n.\end{aligned}$$

We see that all the conditions of Corollary 1 are satisfied. Hence, there exists a smallest semigroup, $\{S_G(t)\}_{t\geq 0}$, generated by the extension G of $A + B$.

5. Conclusion

The discrete growth-fragmentation equation has been considered in this paper. The discrete growth-fragmentation model is closely related to its continuous analogue and is obtained by replacing: the continuous size variable x with the discrete index n; the integrals with sums, and the derivatives with differences. Note also that in the continuous settings the growth-fragmentation equation might require some boundary conditions, this is not necessary in the discrete case. The model (7) has many applications in natural sciences, e.g. in cells division by fission; in protein polymerization; in neuron networks; in packeted data transmission protocols.

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