Using Critiques As A Strategy For Improving The Proofs Of The Equivalence Of Ring Theoretic Concepts

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Abstract

The aim of the study was to investigate if critiques of proofs presentations by the class and the instructor had an impact on writing proofs for the equivalence of definitions of ring theoretic concepts. For a ring A and an ideal J of A, ring-theoretic concepts are defined describing a certain property of J. In the cases that we consider, the property of J has an equivalent definition in terms of the elements in the ring A/J. Eight problems involving equivalence type relations were selected for research (see Appendix). Students worked in pairs on the problem allocated to them to prove the equivalence of two definitions for a particular concept. Each pair of students presented their proof during the weekly tutorial class, one proving the sufficient ("if") part and the other the necessary ("only if") part. The instructor and the students critiqued these proofs, after which the students were instructed to improve their proofs based on the critiques. Both the original and the improved version of the proofs were submitted to the instructor for grading. All the students were then given an unannounced test in which they were instructed to do any two of the eight problems they had worked on. Of the 16 students, 14 attempted their own problem and one new problem in the test. Three weeks later one of the eight problems was selected by the instructor as a final examination question for the course. The Seldon and Seldon (2009) theoretical framework was used to look at the logical construction path, the hierarchical, the rhetorical and the problem solving aspects of the proof. An 8-point marking rubric based on the framework was drawn up and used to grade the proofs. The results indicate that students were able to carry-over some knowledge and experience of proof development from the critiqued class presentations to the test when the question was familiar (first test question), but they were not so successful when they had to develop a proof for a less familiar problem (second test question and the exam question).

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USING CRITIQUES AS A STRATEGY FOR IMPROVING THE PROOFS OF THE EQUIVALENCE OF RING THEORETIC CONCEPTS

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Abstract: The aim of the study was to investigate if critiques of proof presentations by the class and the instructor had an impact on writing proofs for the equivalence of definitions of ring theoretic concepts. For a ring A and an ideal J of A, ring- theoretic concepts are defined describing a certain property of J. In the cases that we consider, the property of J has an equivalent definition in terms of the elements in the ring A/J. Eight problems involving equivalence type relations were selected for research (see Appendix). Students worked in pairs on the problem allocated to them to prove the equivalence of two definitions for a particular concept. Each pair of students presented their proof during the weekly tutorial class, one proving the sufficient ("if") part and the other the necessary ("only if") part. The instructor and the students critiqued these proofs, after which the students revised their proofs based on the critiques. Both the original and the improved version of the proofs were submitted to the instructor for grading. All the students were then given an unannounced test in which they had to do any two of the eight problems they had worked on. Of the 16 students, 14 attempted their own problem and one new problem in the test. Three weeks later one of the eight problems was selected by the instructor as a final examination question for the course. The Seldon and Seldon (2009) theoretical framework was used to look at the logical construction path, the hierarchical, the rhetorical and the problem solving aspects of the proof. An 8-point marking rubric based on the framework was drawn up and used to grade the proofs. The results indicate that students were able to carry-over some knowledge and experience of proof development from the critiqued class presentations to the test when the question was familiar (first test question), but they were not so successful when they had to develop a proof for a less familiar problem (second test question and the exam question). The research is significant in that it (a) highlights the difficulties students have in writing proofs of the equivalence of definitions of ring theoretic concepts; (b) presents a teaching strategy that should lead to improvements in proof writing, student participation and communication. Further research is necessary to find strategies that can maintain the initial gains.

Keywords: abstract algebra, ring theory, ideal, quotient ring, equivalent definition, proof, mathematics education

MSC: Subject Classification 150102

1. Introduction

Definitions are fundamental in mathematics and some of the roles attributed to them include: (1) to introduce the objects of a theory and to convey the characterizing properties of concepts (Mariotti & Fischbein, 1997); (2) to constitute the fundamental components for the formation of concepts (Klausmeier & Feldman, 1975; Sowder, 1980; Vinner, 1991); (3) to establish the foundation for proofs and problem solving (Moore, 1994; Weber, 2002); (4) to create uniformity in the meaning of concepts (Borasi, 1992); and (5) to supply the language, the verbal expression, for the steps in a proof ,that is, how it begins, how it ends, and how the beginning is linked to the ending by rules of logic and a definition, axiom or theorem (Moore, 1994).

Most mathematicians share the view that a proof is most valuable when it leads to understanding, helping them think more clearly and effectively about mathematics (Rav,1999; Manin,1992,1998; Thurston,1994) and that proofs are the "mathematician's way to display the mathematical machinery for solving problems and to justify that a proposed solution to a problem is indeed a solution"(Rav,1999,p.13). A primary reason that proofs are presented to undergraduates is to convince and explain (e.g.,Hersh, 1993).

Selden and Selden (1995) call the readings to determine the correctness of mathematical proofs and the mental processes associated with them "validations of proof". The ability to validate proofs is a crucial skill for mathematicians, mathematics majors and teachers of mathematics but appears to be very challenging (e.g., Selden & Selden, 2003; Alcock and Weber, 2005; Powers, Craviotti, and Grassl, 2010). Principles and Standards for School Mathematics (NCTM,2000) recommends that teachers should discuss the logical structure of students' arguments and assist students in critiquing others' arguments. In deciding whether an argument constitutes a proof, a student or teacher should not only assess whether the proof techniques employed were acceptable to the mathematical community at large but also whether they were understood by the community in which the proof was situated (e.g.,Stylianides, 2007).

Undergraduate mathematics majors need to validate proofs reliably both to check the validity of the mathematical arguments that they produce and to extract conviction from the proofs that are presented to them in their lectures and textbooks (e.g., Selden & Selden, 1995; Weber, 2004; Pfeiffer, 2009).

2. Review of the Literature

Research on ring-theoretic concepts is scarce, with the majority of the articles (e.g. Dubinsky and Leron, 1995; Almeida,1999) focussing on group theory and, as a result, little is known about how students learn the basic ideas of ring theory. Among the conceptual ideas shared with groups are binary operations and associativity, but the current literature does not definitively address the question of how students come to understand a formal algebraic structure lacking one or more properties of fields.

Among the articles located were the following:

Cook (2014) addresses student learning of the concepts of zero-divisor and elements with no multiplicative inverses in rings.

Simpson and Stehlikova (2006) did a case study which examined how one student apprehended the commutative ring $\mathbb{Z}_{99}$.

3. Theoretical Framework

In their study of 109 research-active mathematicians, Inglis, Mejía-Ramos, Weber, and Alcock (2013) found further evidence that there is no universal agreement about what constitutes a valid proof. They concluded that mathematicians' standards of validity differ. To describe what it means to write proofs, this study draws on the work of Selden and Selden (2009) as a framework. Selden and

Selden define five aspects of a proof that they claim must be attended to when constructing proofs with meaning:

1. The Hierarchical Structure, which includes knowing what the proof has to accomplish and coordinating any sub-proofs or constructions, including lemmas.
2. The Construction Path, which is the means for actually creating the proof (as distinct from the way the proof is written for publication). The description of the construction path relies on the concept of an idealized prover who, as Selden and Selden (2009) described, "never erred or followed false leads" (p.340).
3. The Proof Framework, which encompasses the conventions of proving things in mathematics but does not require understanding the meaning of any of the terms. For example, it includes the logical structure of different types of proofs (direct, contradiction, and contrapositive), as well as the concepts of hypotheses and conclusions.
4. The Formal- Rhetorical part of a proof, which requires primarily behavioural knowledge to complete. As Selden and Selden (2009, p.344) argue," it is not important that a student be able to articulate such behavioural knowledge, it is important that he/she can act on it". It includes the ability to do algebraic and technical symbolic manipulations within the structure of the proof system such as those required to deal with quantification or logical implication. This behavioural knowledge includes knowing that if a theorem says, "For all real numbers" then the proof should start by introducing an arbitrary real number, "Let x be a real number" (Selden and Selden, 2009, p. 343).
5. The Problem-Centered part of the proof, which includes determining the key idea(s) and coordinating aspects of the proof (especially if they include nonstandard argument structures), and requires what Selden and Selden (2009, p.334) call "conceptual knowledge, mathematical intuition, and the ability to bring to mind the 'right' resources at the 'right time'".

Based on the Selden and Selden (2009) framework, eight common criteria were identified. The eight criteria together with their contributing weight to the mark allocation for a proof are summarized in Table 1. The line number refers to the line in the proof submitted by the students.

Table 1 Criteria and mark allocation

Criteria	Mark allocation	Line number
1. The hypothesis is written as a complete sentence in which the words "for all" and "for some" are used appropriately. (Seldon and Seldon , No 2, 3, 4)	1 or 0	
2. The student knows how to start a proof (direct, contradiction, or contrapositive). For example, if a theorem says "For all real numbers…" then the proof should start by introducing an arbitrary real number, "Let x a real number….."The student can connect this to the hypothesis. (Seldon and Seldon , No 1, 2, 4 , 5)	2 or 1 or 0	
3. The hypothesis is correctly applied. (Seldon and Seldon , No 2, 5)	1 or 0	
4. Definitions are correctly translated from the ring to the quotient ring e.g. square roots of elements, zero divisor, nilpotent element, invertible elements, Boolean ring. (Seldon and Seldon , No 2, 5)	1 or 0	
5. The rules for equality of cosets in A/J are correctly applied (not important to state the rules). (Seldon and Seldon , No 2, 4)	1 or 0	
6. The rules for addition and/or multiplication of cosets in A/J are correctly applied (not important to state the rules). (Seldon and Seldon , No 2, 4)	1 or 0	
7. The statements follow logically from one to the next using precise language and symbols e.g. $\Longrightarrow$ (with no gaps in the reasoning). (Seldon and Seldon , No 2, 3, 4)	2 or 1 or 0	
8. The conclusion is written as a complete sentence in which the words "for all" and "for some" are used appropriately. (Seldon and Seldon , No 3)	1 or 0	

4. Theoretical background

The present study examines students' understanding and application of certain concept definitions in proving ring-theoretic results in an abstract algebra course. Two examples of such concepts and their definitions are as follows (see Appendix):

(a) Concept definition. An ideal J of a ring A is called *primary* iff for all $a, b \in A$, if $ab \in J$, then either $a \in J$ or $b^n \in J$ for some positive integer n.
(b) Concept definition. An ideal J of a ring A is called *semi-prime* iff it has the following property: For every $a \in A$, if $a^n \in J$ for some positive integer n, then necessarily $a \in J$.

The property of J mentioned in both these examples has an equivalent definition in terms of the elements in the quotient ring A/J. For example, (a) every zero divisor in A/J is nilpotent iff J is primary; (b) J is semi-prime iff A/J has no nilpotent elements (except zero).

In the research, students were given the task of proving such equivalent definitions of a property of the ideal J in terms of the elements in the quotient ring A/J. Concepts, such as the ones mentioned above, were unfamiliar to the students.

5. Research Objectives

The main objectives of the study were to improve class participation and to investigate how students prove the equivalence of definitions of non-familiar concepts in which inferences were largely based on definitions of concepts and some elementary properties of A/J, including rules for equality of elements (cosets) in A/J and rules for adding and multiplying cosets. Thus proof was used in this research to assess students' understanding of unfamiliar concepts and their equivalent definitions in terms of elementary properties of the elements of A/J. This was afforded by the problems selected for this research (see Appendix).

6. Participants

The participants in this research were third-year Abstract Algebra students ($n = 16$) at a university in the Western Cape province of South Africa. All students had passed a semester of introductory Abstract Algebra that included an introduction to proofs, a semester of Linear Algebra as well as a course in Group Theory covering the material presented in the first sixteen chapters of the prescribed textbook by Pinter, C.C (1990), A Book of Abstract Algebra (Second Edition).

7. Research Methodology

One of eight problems on ring-theoretic statements from Pinter, C.C (1990) was allocated randomly to two students. For a ring A, and an ideal J of A, the problems all try to answer the question of how properties of J transfer to properties of the quotient ring A/J. In all eight problems, the property of J has an equivalent description in terms of the elements in A/J. Students were then given the task of proving this equivalence.

The sixteen students in the class were paired. Each pair of students was allocated a problem. The students were given time over a weekend to complete and hand in their proof for marking. In the next three weeks, time was given in the tutorial class for each pair of students to present their proof to the rest of the class on the chalkboard. The instructor and the class critiqued the arguments presented and students were encouraged to comment on the structure of the proof, style of writing (e.g. the statements follow logically from one to the next using precise language and symbols, writing complete sentences with the correct use of quantifiers "for all" and "for some" and leading to a conclusion), the correct use of definitions and assumptions, validity of the arguments presented or whether there were any gaps in the reasoning.

Following the validation process, the presenting group was given an opportunity to rethink the concepts and proof steps and then to resubmit their proof for marking. Both the original proofs and the resubmitted proofs were retained for use in the research.

An unannounced test with the instruction that they could do any two of the eight problems that they worked on, including their own problem, was given the week following the last presentation. Three weeks later one of the eight problems was selected by the instructor as a final examination question for the course. A summary of the procedure is presented in Figure 1.

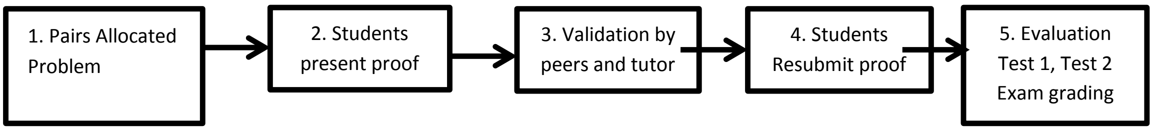

Figure 1. Procedure used for proof development and evaluation

8. Assessing the validity of the proofs based on the theoretical framework

In what follows we give examples of our expectation of a proof showing the equivalence of two definitions of a certain ring-theoretic concept. We used Exercises E3 and G3 in the Appendix. We focus more on the structure of the proof.

E3. A/J is a Boolean ring iff $x^2 - x \in J$ for every $x \in A$. (A ring S is called a Boolean ring iff $s^2 = s$ for every $s \in S$.)
Proof: Assume A/J is a Boolean ring [$X^2 = X$, for every $X \in A/J$] [Hypothesis].
Let x be any fixed but arbitrarily chosen element of A [Comes from knowing what to prove].
Consider the element $J + x \in A/J$ [Connection to the hypothesis].
By the hypothesis, $(J + x)^2 = J + x$ [Application of the hypothesis & definition of Boolean ring].
Therefore $J + x^2 = (J + x)^2 = J + x$ [By the rule for multiplication of cosets in A/J].
$\Rightarrow x^2 - x \in J$ [By the rule for equality of cosets in A/J].
This proves that $x^2 - x \in J$ for every $x \in A$ [Conclusion].
Conversely, assume that $x^2 - x \in J$, for every $x \in A$ [Hypothesis].
Let X be any fixed but arbitrarily chosen element of A/J [Comes from knowing what to prove].
Then $X = J + a$ for some $a \in A$ [Connection to the hypothesis].
By the hypothesis, $a^2 - a \in J$ [Application of the hypothesis].
Hence $J + a^2 = J + a$ [By the rule for equality of cosets in A/J].
Therefore $X = J + a = (J + a)^2$ [By the rule for multiplication of cosets in A/J].
This proves $X^2 = X$.
Since X was arbitrarily chosen, A/J is a Boolean ring [Conclusion].

G3. An ideal J of a ring A is called primary iff for all $a, b \in A$, if $ab \in J$, then either $a \in J$ or $b^n \in$ J for some positive integer n. Prove that every zero divisor in A/J is nilpotent iff J is primary.

Assume J is a primary ideal [Hypothesis].
Let X be any fixed but arbitrarily chosen zero divisor in A/J [Comes from knowing what to prove].

Then $X = J + a$ where $a \notin J$ and there exists $J + b \in A/J$, where $b \notin J$ such that $(J + a)(J + b) = J$ [Definition of zero divisor in A/J].
Therefore $J = (J + a)(J + b) = J + ab$ [By the rule for multiplication of cosets in A/J].
$\Rightarrow ab \in J$ [By the rule for equality of cosets in A/J].
Therefore $ba \in J$ since A is a commutative ring [Using assumptions].
It follows that $b \in J$ or $a^n \in J$, for some positive integer n, since J is a primary ideal [Application of the hypothesis].
But $b \notin J$ and so $a^n \in J$. Therefore $J + a^n = J$ [By the rule for equality of cosets in A/J].
Now $J = J + a^n = (J + a)(J + a)...(J + a)$ n times $= (J + a)^n$ [By the rule for multiplication of cosets in A/J].
This shows that $X = J + a \in A/J$ is nilpotent [Definition of nilpotence of $J + a \in A/J$].
Since X is an arbitrarily chosen zero divisor, we conclude that every zero divisor is nilpotent [Conclusion].

Conversely, assume every divisor of zero in A/J is nilpotent [Hypothesis].
Let a and b be any two arbitrarily chosen elements in A with $ab \in J$
[Comes from knowing what to prove].
This is true if either $a \in J$ or $b \in J$; so we assume $a \notin J$ and $b \notin J$ [Using assumptions].
Since $ab \in J$, it follows that $J + ab = J$ [By the rule for equality of cosets in A/J].
But then $(J + a)(J + b) = J + ab = J$ [By the rule for multiplication of cosets in A/J].
Since $a \notin J$ and $b \notin J$, it follows that $J + a \neq J$ and $J + b \neq J$
[By the rule for equality of cosets in A/J].
Therefore $J + b$ is a divisor of zero in A/J [Definition of zero divisor in A/J].
$\Rightarrow (J + b)^n = J$ for some positive integer n [Application of the hypothesis].
Now $J = (J + b)^n = (J + b)(J + b)...(J + b)$ for n factors $= J + b^n$
[By the rule for multiplication of cosets in A/J].
$\Rightarrow b^n \in J$, for some positive integer n [By the rule for equality of cosets in A/J].
This proves that J is a primary ideal [Conclusion].

It can be seen by comparison that the two model proofs of the statements in E3 and G3 are very similar in structure.

9. Class presentations by students

The presentations were made in the lecture room, which had two columns of desks arranged in 6 rows, each having 3 to 4 desks. The chalkboard was along the width of the lecture room on the front wall. The observer sat at the back with a good overview of the proceedings in the class and took notes and recorded transcripts of the discourse. There were occasional comings and goings as students arrived late or left to take calls and some side conversations between students preparing for their presentations.

Triangulation was done by going over the recorded class observations, recorded conversations and written notes made during the presentations. These were double checked with the instructor after the lecture.

Each pair of students came to the chalkboard and wrote out their proof, one presented the direct proof and the other member of the pair presented the converse. A presentation for the proof of Problem G3 ran as shown in Table 2. The step numbering was added by the researcher to make it easy for reference in the discussion that follows.

The proofs presented were highly symbolic and punctuated with rhetorical questions as the instructor probed students recall and understanding of previously presented material. The discourse used can be categorised in terms of the highly technical vocabulary used e.g. nilpotent, zero divisor, coset, arbitrary elements, etc. the assumptions, e.g. suppose every element in A/J is its own negative.., and definitions, e.g. since J is a primary ideal $J + a^n = ...$

Table 2

Direct proof presented by Student A.	Converse presented by Pair Partner – Student B.
1. Suppose J is a primary ideal. 2. Let $J + a \in A/J$ be a zero divisor 3. then there exists $b \in A$ such that $(J + a)(J + b) = J,\ b \notin J$ 4. $(J + a)(J + b) = J + ab = J$ 5. $\Rightarrow ab \in J$ 6. $a^n \in J$ for some positive integer n since J is a primary ideal 7. $J + a^n = (J + a)(J + a)...(J + a) = (J + a)^n = J$ 8. $\therefore J + a \in A/J$ is nilpotent	1. Suppose every zero divisor in A/J is nilpotent. 2. Let $a, b \in A$ and $ab \in J$ 3. $(J + a)(J + b) = J + ab = J$ 4. If $a \notin J$ then $J + b$ is a zero divisor in A/J 5. $\therefore (J + b)^n = J$ 6. but $(J + b)^n = (J + b)\ (J + b).....(J + b)$ n times 7. $= J + b^n = J$ 8. $\Rightarrow b^n \in J$ 9. $\therefore J$ is a primary ideal.

Student A then goes over the direct proof verbally and the class participates by giving their comments.

Prof: "Read the problem again. In line1 should you use suppose or assume?"

Stud A: "Assume allows us to prove something else. If we suppose … then we can end with a contradiction"

Prof: "Some people use "suppose" interchangeably with "assume". So it is not wrong to say "suppose". But you are right, we use "suppose" when we do proofs by contradiction."

Student A decides to amend line 1 and write "Assume that …."

Prof: "Is your definition of zero divisor in lines 2 and 3 correct?"

Student A: "It must be $a \notin J$".

Prof: "Can you explain how you come to conclude that $a^n \in J$ for some positive integer n using the fact that J is a primary ideal" in line 6.

Student A is unable to explain the reason for this conclusion. The question is directed to the class.

Prof: "Look at the definition of a primary ideal. What can you conclude from line 5?"

Student B: "either $a \in J$ or $b^n \in J$ for some positive integer n."

Prof: "But this is not what we want, is it? What do we want?"

Student A:"either $b \in J$ or $a^n \in J$ for some positive integer n."

Prof: "Are there any missing steps between lines 5 and 6?"

Student C: (comes to the board and writes)

$(J+a)(J+b)$

$= (J+b)(J+a)$

$= ba$

$\therefore\ (ba)^n \in J$

So $b \in J$ or $a \in J$

Prof: "Is this correct?" No response from the class.

Prof: “It is incorrect but there is a hint in what student C wrote. If we knew that ba$\in$ J then what could we conclude from this?”

Student A:” $b \in J$ or $a^n \in J$, by applying the definition of primary ideal.”

Prof: “What do we know about the ring A? What assumptions can we make about A? Look at the statement of the problem.”

Student D: “A is a commutative ring.”

Student D:” $ab = ba$”

Prof: “So where is ba?”

Student D:” $ba \in J$.”

Prof: “What can we now conclude?”

Student A:” $b \in J$ or $a^n \in J$, by applying the definition of primary ideal.”

Prof: “What do we know about b?”

Student B: “$b \notin J$ and $\therefore a^n \in J$”

Prof:” Look at the rest of the steps. I suggest that line 7 be split into two lines to make the reading clearer.”

Prof: “Write $J = J + a^n$, from line 6 by the rule for equality of cosets

$= (J + a)(J + a)\ldots(J + a) = (J + a)^n$,by the rule for multiplying cosets.”

The main points emerging from this discussion are that the proof assumes a thorough knowledge of the mathematics involved as well as methods of proof. In lines 2 and 3, Student A incorrectly gives the definition of a zero divisor in A/J. In line 6 the incorrect application of a definition leads to a gap in the logical reasoning. This could be remedied by the knowledge and application of assumptions in the statement of the problem, viz., A is a commutative ring.

A similar discussion followed the converse proof presented by Student B.

Student B started by changing the” Suppose” to “Assume” in line 1.

Prof: “Explain the reasons for your line 3.”

Stud B:”The left equality is multiplying two cosets and the right equality is by the rule for equality of cosets.”

Prof: “Can you explain Line 4.”

Stud B: No response.

Prof: “Look at line 2. What happens if $a \in J$ or $b \in J$?”

Stud C:“If $a \in J$ or $b \in J$ then $ab \in J$.”

Prof: “So what assumptions can we make about a and b?”

Student C: ”$a \notin J$ and $b \notin J$”.

Prof: “What do we conclude from this?”

Stud B:”$J \neq J + a$ in A/J and $J + b \neq J$ in A/J”

Prof: “What can we now conclude from line 3?”

Student B: “$J + b$ is a divisor of zero in A/J”.

Prof: “Explain your statement in line 5.”

Student B: “$J + b$ is nilpotent by the hypothesis; so $(J + b)^n = J$ for some positive integer n.”

Prof: “For the remaining lines, add reasons.”

The converse proof presented by Student B required the application of different methods of proof viz., proof by cases. In a few statements conclusions are written without giving any reasons, for example, in line 5, the fact that $J + b$ is nilpotent follows from the hypothesis.

As an illustration we now apply the rubric to the proof submitted by student A and student B. The line number refers to the line in the proof submitted by the students (see Tables 3 and 4).

Table 3: Group 1 (Students A and B)(Direct Proof)

Criteria	Mark allocation	Line number
1. The hypothesis is written as a complete sentence in which the words "for all" and "for some" are used appropriately. (Seldon and Seldon , No 2, 3, 4)	1	1
2. The student knows how to start a proof (direct, contradiction, or contrapositive). For example, if a theorem says "For all real numbers…" then the proof should start by introducing an arbitrary real number, "Let x a real number…..."The student can connect this to the hypothesis. (Seldon and Seldon , No 1, 2, 4, 5)	1	2
3. The hypothesis is correctly applied. (Seldon and Seldon , No 2, 5)	0	6
4. Definitions are correctly translated from the ring to the quotient ring e.g. square roots of elements, zero divisor, nilpotent element, invertible elements, Boolean ring. (Seldon and Seldon , No 2, 5)	1	3
5. The rules for equality of cosets in A/J are correctly applied (not important to state the rules). (Seldon and Seldon , No 2, 4)	1	5 , 7
6. The rules for addition and/or multiplication of cosets in A/J are correctly applied (not important to state the rules). (Seldon and Seldon , No 2, 4)	1	4, 7
7. The statements follow logically from one to the next using precise language and symbols e.g. $\Longrightarrow$ (with no gaps in the reasoning). (Seldon and Seldon , No 2, 3, 4)	1	5, 6
8. The conclusion is written as a complete sentence in which the words "for all" and "for some" are used appropriately. (Seldon and Seldon , No 3)	0	N/A

Table 4: Group 1 (Students A and B)(Converse Proof)

Criteria	Mark allocation	Line number
1. The hypothesis is written as a complete sentence in which the words "for all" and "for some" are used appropriately. (Seldon and Seldon , No 2, 3, 4)	1	1
2. The student knows how to start a proof (direct, contradiction, or contrapositive). For example, if a theorem says "For all real numbers…" then the proof should start by introducing an arbitrary real number, "Let x a real number….". The student can connect this to the hypothesis. (Seldon and Seldon , No 1, 2, 4 , 5)	1	2
3. The hypothesis is correctly applied. (Seldon and Seldon , No 2, 5)	1	5
4. Definitions are correctly translated from the ring to the quotient ring e.g. square roots of elements, zero divisor, nilpotent element, invertible elements, Boolean ring. (Seldon and Seldon , No 2, 5)	0	4
5. The rules for equality of cosets in A/J are correctly applied (not important to state the rules). (Seldon and Seldon , No 2, 4)	1	3 , 8
6. The rules for addition and/or multiplication of cosets in A/J are correctly applied (not important to state the rules). (Seldon and Seldon , No 2, 4)	1	3, 6
7. The statements follow logically from one to the next using precise language and symbols e.g. $\Longrightarrow$ (with no gaps in the reasoning). (Seldon and Seldon , No 2, 3, 4)	1	3, 4
8. The conclusion is written as a complete sentence in which the words "for all" and "for some" are used appropriately. (Seldon and Seldon , No 3)	1	9

Total: $\frac{13}{20}$

After the students had presented their proofs to the rest of the class, they were expected to reflect on their proofs and to relook the steps required in the proving process and to resubmit a revised proof to the instructor for marking. Such actions often occur in constructing the formal-rhetorical part of a proof (see Table 5).

10. Resubmitted proof

Table 5

Direct proof	Converse
1. Assume J is a primary ideal	1. Assume every zero divisor in A/J is nilpotent.
2. and let $J + a \in A/J$ be a zero divisor ,	2. Let $a, b \in A$ *and* $ab \in J$
3. ,then there exists $J + b \in A/J$, $b \notin J$ such that $(J + a)(J + b) = J$	3. $(J + a)(J + b) = J + ab = J$
4. $(J + a)(J + b) = J + ab = J$	4. If $a \in J$ or $b \in J$ then $ab \in J$
5. $\Longrightarrow ab \in J$	5. So we can assume $a \notin J$ and $b \notin J$
6. Since A is a commutative ring then $ba \in J$	6. $\Longrightarrow J + a \neq J$ *and* $J + b \neq J$
7. and $b \in J$ or $an \in J$ since J is a primary ideal for some positive integer n	7. therefore $J + b$ is a zero divisor in A/J
8. $b \notin J$ so $an \in J$	8. $\Rightarrow (J + b)n = J$
9. $\Rightarrow J + an = J$	9. but $(J + b)n = (J + b)(J + b) \ldots (J + b)$ n times
10. $J + an = (J + a)(J + a) \ldots (J + a) = (J + a)n = J$	10. $= J + bn = J$
11. $\therefore J + a \in A/J$ is nilpotent	11. $\Longrightarrow b^n \in J$
	12. $\therefore J$ is a primary ideal.

Although the suggested changes in the validation of the initial proof have to a large extent been incorporated in the re-submitted proof, the readability can be improved by giving reasons for conclusions and writing full sentences. For example, in the converse proof, $J + b$ is a zero divisor in A/J by line 3. Line 8 should read " $\Rightarrow (J + b)^n = J$ for some positive integer n; because we assume all divisors of zero are nilpotent. Line 3 should start with the implication symbol " $\Longrightarrow$". The rubric can now be applied to the re-submitted proof as was done with the original proof.

The marks obtained for the resubmitted proof by all eight groups were higher than for the original proofs. This begs the question whether the validation process led to an improved understanding of proving statements of "if, and only if" type. It was for this reason that the students were given an unannounced test with the instruction that they could do any two of the eight problems that they worked on, including their own problem. As expected, 14 of the 16 students attempted their own problem and one new problem. Three weeks later one of the eight problems was selected by the instructor as a final examination question for the course. In the next section we analyse the marks obtained by the students.

11. Data Analysis

Sixteen of the seventeen third-year mathematics class attending the Abstract Algebra module were part of this intervention aimed at using critiques to develop skills in proof writing. One student dropped out and did not attempt the test or exam questions. Due to the small group size and the non-normality of the data, non-parametric tests are reported. Table 6 presents the descriptive statistics of the marks obtained in the assignments, the test questions and related exam question. All proofs were evaluated by using the rubric explained in Table 1.

Table 6: Descriptive statistics of the rubric marks (as %) obtained

	n	mean	Std dev	95% CI mean	median	95% CI median	min	max
Assignment	16	46.88	16.46	38.11 - 55.64	50.00	31.25 - 54.52	25.00	75.00
Resubmitted assignment	16	64.84	12.68	58.09 - 71.60	62.50	56.25 - 69.28	50.00	87.50
Test problem 1	16	54.30	17.34	45.06 - 63.54	56.25	47.74 - 68.75	6.25	75.00
Test problem 2	16	44.92	21.20	33.63 - 56.21	53.13	22.74 - 62.50	6.25	68.75
Exam question G1	16	29.69	24.31	16.73 - 42.64	21.88	12.50 - 56.25	0.00	62.500

For the initial class assignment done in pairs, students scored on average 47% (median=50%, std dev=16.46), whereas the re-submission mark after critique was 65% (median=63%, std dev=12.68) (see Table 6). A significant improvement is noticeable (Wilcoxon signed ranks test=68; p=0.0001).

Students were asked to complete two such questions in the first test and fourteen of the sixteen (87.5%) attempted the proof that they had developed during the assignment and critique sessions. Twelve students (75%) passed the first test question. The second test question was selected from the pool of questions that other students had developed as part of their training. Nine of the students (56%) passed the second test question, which confirms that most students were able to transfer the knowledge on how to construct a proof from one question to another.

For the first test problem students obtained an average of 54% (median= 56%, std dev=17.34) with twelve of the sixteen passing this question. For the second test problem students scored on average 45% (median=53%, std dev=21.20) with nine of the twelve obtaining more than 50% for the question. No difference was seen between the mark obtained for the first test problem (more familiar question) when compared to the resubmitted class assignment's mark (Wilcoxon signed ranks test=23; p=0.2048), but students scored significantly less in the second test question (less familiar question) when compared to the resubmitted class assignment's mark (Wilcoxon signed ranks test=31; p=0.0295).

The final exam paper was written a few weeks later which contained one question where students had to develop a proof. Again this question was selected from the original questions provided to students at the start of the intervention. This exam question was answered by all the students and the average was 30% (median=22%, std dev=24.31) but only 6 students (37.5%) passed this question. The mark obtained in the exam was significantly lower when compared to the first test question's mark (Wilcoxon signed ranks test=55; p=0.0030) as well as the second test question's mark (Wilcoxon signed ranks test=34; p=0.0129).

Table 7 shows the descriptive statistics of the marks separately for those who failed or passed the exam question. When considering the 95% confidence intervals (CI) for the median of the exam pass and exam fail groups; the initial assignment submission, the resubmission and the test marks do not differ as these confidence intervals between the two groups overlap.

Students who passed the exam question, obtained higher median percentages in both tests compared to those who failed the exam question (Test1=69% and Test2=62% for those who passed; Test1=53% and Test2=41% for those who failed the exam)(see Table 7).

Table 7: Descriptive statistics for students who failed or passed the exam question

	Descriptive statistics of marks for those who failed or passed the exam							
	Failed Exam				Passed Exam			
	N	Mean	Median	95% CI median	N	Mean	Median	95% CI median
Assignment	10	48.75	46.88	31.25 - 69.06	6	43.75	50.00	25.00 - 60.07
Resubmitted assignment	10	67.50	62.50	59.22 - 84.53	6	60.422	59.38	50.00 - 77.61
Test problem 1	10	50.63	53.13	43.75 - 62.50	6	60.42	68.75	36.10 - 68.75
Test problem 2	10	39.38	40.63	18.75 - 56.25	6	54.17	62.50	15.96 - 68.75
Exam question G1	10	13.125	12.50	0.00 - 22.34	6	57.29	56.25	51.21 - 62.50

The individual marks of students who *passed* and *failed* the exam are shown in Figure 2. From Figure 2 it can be seen that in general student marks improved from the initial assignment mark to the re-submitted assignment's mark. Except for one student, all the other students who failed the exam, scored lower percentages in the exam question than in the two test questions. When focusing on those who passed the exam, one student performed better in the exam compared to all the prior marks. In general, for the group who passed, the exam mark was slightly lower than the test marks.

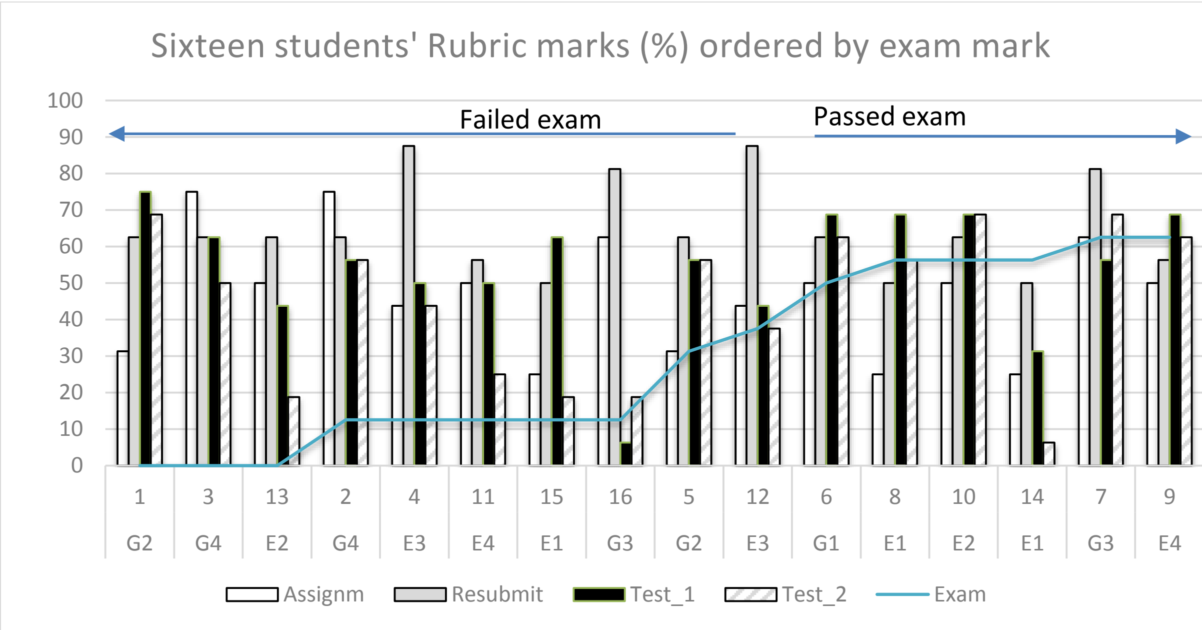

Figure 2: Marks ordered by exam mark

When inspecting the results of the Spearman correlation coefficient (Table 8) it can be seen that only two of the marks were significantly correlated. The mark obtained in the first test problem is significantly correlated with the mark obtained in the second test question (Spearman coefficient=0.773, p=0.0004). This suggests that students were able to carry over knowledge on how to develop a proof from one problem to another similar type of problem in the test.

None of the marks obtained in earlier evaluations were correlated with the mark obtained in the exam question.

Table 8: Spearman correlations of the marks obtained

Spearman Correlation Coefficients, N = 16 Prob > |r| under H0: Rho=0				
	Resubmitted assignment	Test1	Test2	Exam
Assignment	0.44777 0.0820	-0.10419 0.7010	0.21533 0.4232	-0.15119 0.5762
Resubmitted assignment	1.00000	-0.31427 0.2358	0.17693 0.5121	-0.16410 0.5437
Test problem 1		1.00000	0.77294 0.0004*	0.16059 0.5524
Test problem 2			1.00000	0.35132 0.1821

*Significant at a 1% level of significance

When considering each of the rubric criteria (Table 1) student groups could score between zero and two for each criteria. Each criteria is shown for the initial assignment mark followed by the resubmission mark for that criteria (a total of eight aspects are shown in Figure 3). It can be seen that criteria 5 and 6 were mostly achieved by the students. In criteria 7 students scored the lowest with no improvement in the resubmitted score for some questions (E1, E2, E4, G1 and G4). Criteria 8 also showed no improvement in the resubmitted score for a few questions (E1, E4 and G2).

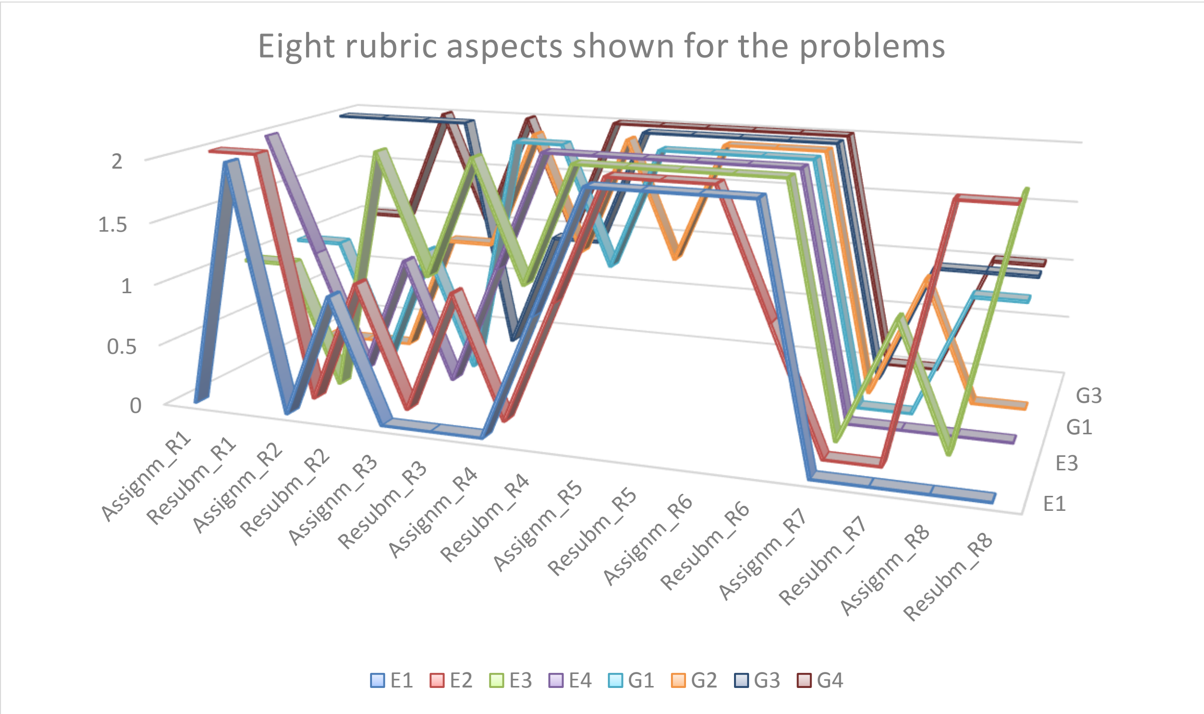

Figure 3: Eight rubric criteria shown for each problem (initial assignment and resubmission)

12. Significance and directions for future study

Proofs in abstract algebra have well defined structures that follow a set of logical steps that link with one another, starting from hypotheses and leading to a conclusion by only using axioms, definitions, previously proved results and rules of inference. Research has shown how students' inability to start on a proof was symptomatic of many other difficulties and that they relied on memorizing proofs because they had not understood what a proof is nor how to write one (Moore, 1994). Research also revealed that both students and teachers of mathematics have difficulty in accurately determining whether an argument constitutes a valid proof (e.g. Alcock & Weber, 2005; Selden & Selden, 2003). In this research students learned to do proofs of mathematical statements that required only short deductive proofs; thus preparing them for the transition to more advanced proofs that require more complex cognitive processes. In addition, an attempt was made to actively engage students in the validation of, not only their own proof, but also the proofs of their peers. This was guided by comments made by the instructor as the process of validation unfolded.
This research has laid the foundation for a follow-up study which will attempt to answer the following questions:

- What is the impact of collaborative work on proving a mathematical statement?
- What is the impact of the validation process on proving a mathematical statement?

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13.1. Competing interests

The authors declare that they have no financial or personal relationship(s) which may have inappropriately influenced them in writing this article.

13.2. Authors'contributions

The research was done with a group of third year abstract algebra students. The lecturer for the course, R. L. Fray, as the mathematician in the group of researchers, was responsible for the mathematical aspects of this paper. As an expert in statistics, R. Blignaut did the statistical analysis in the paper. As the education specialist, O. Sheikh was responsible for the educational aspects of the paper.

The manuscript went through several drafts with all three authors contributing evenly to the final write-up.

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15. Appendix

E. Properties of Quotient Rings A/J in Relation to Properties of J (Pinter, p 197)

Let A be a ring and J an ideal of A. Prove each of the following:

1. Each element of A/J has a square root iff for every $x \in A$, there is some $y \in A$ such that $x-y^2 \in J$.
2. Every element of A/J is its own negative iff $x+x \in J$ for every $x \in A$.
3. A/J is a Boolean ring iff $x^2-x \in J$ for every $x \in A$. (A ring S is called a Boolean ring iff $s^2 = s$ for every $s \in S$.)
4. Every element of A/J is nilpotent iff J has the following property: for every $x \in A$, there is a positive integer n such that $x^n \in J$.
   Note this is problem 5 in exercise E on page 197.

G. Further Properties of Quotient Rings in Relation to Their Ideals (Pinter, p 198)

Let A be a ring and J an ideal of A. In parts 1-3 assume that A is a commutative ring with identity.

1. Prove that A/J is a field iff for every element $a \in A$, where $a \notin J$, there is some $b \in A$ such that $ab - 1 \in J$.
2. Prove that every nonzero element of A/J is either invertible or a divisor of zero iff the following property holds, where $a, x \in A$: For every $a \notin J$, there is some $x \notin J$ such that either $ax \in J$ or $ax - 1 \in J$.
3. An ideal J of a ring A is called *primary* iff for all $a, b \in A$, if $ab \in J$, then either $a \in J$ or $b^n \in J$ for some positive integer n. Prove that every zero divisor in A/J is nilpotent iff J is primary.
4. An ideal J of a ring A is called *semi-prime* iff it has the following property: For every $a \in A$, if $a^n \in J$ for some positive integer n, then necessarily $a \in J$. Prove that J is semi-prime iff A/J has no nilpotent elements (except zero).

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