An Operator Norm Independent Solution of Monotone Variational Inclusion Problem in Hilbert Space

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Abstract

In this paper, we introduce a general iterative algorithm that does not require any knowledge of the operator norm for approximating a solution of a split monotone variational inclusion problem which is also a common element of the set of fixed points of a finite family of strictly pseudocontractive mappings. Furthermore a strong convergence theorem for approximating a common solution of a monotone variational inclusion problem and a fixed point problem of a finite family of strictly pseudocontractive mappings, which is also a solution of some variational inequality problems was stated and proved in the frame work of Hilbert spaces

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AN OPERATOR NORM INDEPENDENT SOLUTION OF MONOTONE VARIATIONAL INCLUSION PROBLEM IN HILBERT SPACE

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Abstract: In this paper, we introduce a general iterative algorithm that does not require any knowledge of the operator norm for approximating a solution of a split monotone variational inclusion problem which is also a common element of the set of fixed points of a finite family of strictly pseudocontractive mappings. Furthermore a strong convergence theorem for approximating a common solution of a monotone variational inclusion problem and a fixed point problem of a finite family of strictly pseudocontractive mappings, which is also a solution of some variational inequality problems was stated and proved in the frame work of Hilbert spaces.

Keywords: κ-pseudocontractive; Hilbert space; Split monotone variational inclusion; Fixed point; Bounded linear operator; Strong convergence.

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1. Introduction

Let H be a real Hilbert space and K a nonempty, closed and convex subset of H. A mapping $T : K \to K$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\| \ \ \forall \ \ x, y \in K, \tag{1}$$

$T : K \to K$ is said to be *a contraction* if there exists $L \in (0, 1)$ such that

$$||Tx - Ty|| \leq L||x - y||, \ \ \forall \ \ x, y \in K, \tag{2}$$

and $T : K \to K$ is said to be κ-*strictly pseudocontractive* in the sense of Browder and Petryshyn [4] if for $0 \leq \kappa < 1$,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa\|(I - T)x - (I - T)y\|^2 \ \ \forall \ \ x, y \in K. \tag{3}$$

Clearly (3) is equivalent to

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2 - \frac{1-\kappa}{2}\|(I-T)x - (I-T)y\|^2. \tag{4}$$

See [1,4,21,31]and references therein, for more details on strictly pseudocontrative mappings. The set of fixed points of T is the set $F(T) = \{x \in K : Tx = x\}$.

A bounded linear operator D on H is called strongly positive if there exists $\bar{\delta} > 0$ such that

$$\langle Dx, x\rangle \geq \bar{\delta}||x||^2, \ \ \forall x \in H.$$

Let K be a nonempty closed and convex subset of a real Hilbert space H. The metric projection from H onto K denoted by P_K, is the the map that assigns to each $x \in H$ the unique point $P_K x \in K$ with the property

$$||x - P_K x|| = \inf\{||x - y||, \forall y \in K\}.$$

The following also holds for P_K:
(i). P_K is nonexpansive.
(ii). $\langle y - P_K(x), x - P_K(x)\rangle \leq 0, \ \ \forall x \in H, y \in K.$

Let $f : H \to H$ be a single valued nonlinear mapping and let $M : H \to 2^H$ be a set valued mapping. The Monotone Variational Inclusion Problem (MVIP) is to find $x \in H$ such that

$$0 \in f(x) + M(x), \tag{5}$$

where 0 is the zero vector in H. The set of solutions to the MVIP (5) is denoted by $I(f, M)$. If $f \equiv 0$, then MVIP (5) reduces to the following Variational Inclusion Problem (VIP): find $x \in H$ such that

$$0 \in M(x), \tag{6}$$

For further details on VIP see for example [27] and some of the references therein.
A mapping $T : H \to H$ is said to be
(i) monotone, if

$$\langle Tx - Ty, x - y\rangle \geq 0, \ \ \forall x, y \in H;$$

(ii) α-strongly monotone, if there exists a constant $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \alpha\|x - y\|^2, \ \ \forall x, y \in H;$$

(iii) β-inverse strongly monotone(β-ism), if there exists a constant $\beta > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \beta\|Tx - Ty\|^2, \ \ \forall x, y \in H;$$

(iv) firmly nonexpansive, if

$$\langle Tx - Ty, x - y\rangle \geq \|Tx - Ty\|^2, \ \ \forall x, y \in H.$$

A set valued mapping $M : H \to 2^H$ is called monotone if for all $x, y \in H$ with $u \in M(x)$ and $v \in M(y)$ then

$$\langle x - y, u - v\rangle \geq 0.$$

A monotone mapping M is said to be maximal if the graph of M, denoted by $G(M)$ is not properly contained in the graph of any other monotone mapping, where for multi valued mapping M,

$$G(M) = \{(x, y) : y \in M(x)\}.$$

It is well known that M is maximal if and only if for $(x, u) \in H \times H$, $\langle x - y, u - v\rangle \geq 0$ for all $(y, v) \in G(M)$ implies $u \in M(x)$. The resolvent operator J_λ^M associated with M and λ is the mapping $J_\lambda^M : H \to H$ defined by

$$J_\lambda^M(x) = (I + \lambda M)^{-1}x, \ \ x \in H, \lambda > 0. \tag{7}$$

It is a common knowledge that the resolvent operator J_λ^M is single valued, nonexpansive and 1-inversely monotone (for example see [3]) and the solution of (5) is a fixed point of $J_\lambda^M(I - \lambda f)$, $\forall \lambda > 0$ (see for example [17]). If f is μ-inversely strongly monotone mapping with $0 < \lambda < 2\mu$, then clearly $J_\lambda^M(I - \lambda f)$ is nonexpansive and $I(f, M)$ is closed and convex.

Let H_1 and H_2 be real Hilbert spaces. Let $f_1 : H_1 \to H_1$, $f_2 : H_2 \to H_2$ be inverse strongly monotone mappings and $B_1 : H_1 \to 2^{H_1}$, $B_2 : H_2 \to 2^{H_2}$ be maximal monotone mappings. Let $A : H_1 \to H_2$ be a bounded linear mapping. The Split Monotone Variational Inclusion Problem (SMVIP) is to find $x^* \in H_1$ such that

$$0 \in f_1(x^*) + B_1(x^*) \tag{8}$$

and

$$y^* = Ax^* \in H_2 \ \text{ such that } \ 0 \in f_2(y^*) + B_2(y^*). \tag{9}$$

We shall denote by Ω the solution set of (8) - (9), that is

$$\Omega = \{x^* \in H_1 : 0 \in f_1(x^*) + B_1(x^*) \ \text{ and } \ y^* = Ax^* \in H_2 \ \text{ such that } \ 0 \in f_2(y^*) + B_2(y^*)\}.$$

Moudafi in [23] first introduced the SMVIP (8) - (9) and proposed an iterative method for solving it. In [23], Moudafi noted that the SMVIP is a generalisation of the split fixed point problem, split variational inequality problem, split zero problem and split feasibility problem (see [7–10,15,23–26,29]), which have been studied extensively by many authors and applied to solving many real life problems such as modelling intensity-modulated radiation therapy treatment planning [9,10], modelling of inverse problems arising from phase retrieval and sensor networks in computerised tomography and data compression [6,11].

Suppose $f_1 \equiv 0$ and $f_2 \equiv 0$ in SMVIP (8) - (9), we obtain the following Split Variational Inclusion Problem (SVIP): Find $x^* \in H_1$ such that

$$0 \in B_1(x^*) \tag{10}$$

and

$$y^* = Ax^* \in H_2 \ \text{ such that } \ 0 \in B_2(y^*). \tag{11}$$

Let the solution set of (10)-(11) be denoted be Ω_B.
Byrne *et al.* [7] using the following iterative scheme: for a given $x_0 \in H_1$ the sequence $\{x_n\}$ generated iteratively by;

$$x_{n+1} = J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n), \ \ \lambda > 0,$$

obtained a weak and strong convergence theorem for solving SVIP (10)-(11). Inspired by the work of Byrne *et al.*, Kazmi and Rizvi [16] proposed the following algorithm for approximating a solution of SVIP (10)-(11) which is a fixed point of a nonexpansive mapping S: for a given $x_0 \in H_1$, let the sequences $\{u_n\}$ and $\{x_n\}$ be generated by

$$\begin{cases} u_n = J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n), \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Su_n, n \geq 0, \end{cases} \tag{12}$$

and proved that both $\{u_n\}$ and $\{x_n\}$ converge strongly to $z \in F(S) \cap \Omega_B$.. For more on variational inclusion problem see [18,19].
Shehu and Ogbuisi [28] stated and proved the following theorem for solving SMVIP.

Theorem 1. *Let H_1 and H_2 be two real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator. Let $f_1 : H_1 \to H_1$ be μ-inverse strongly monotone mapping and $f_2 : H_2 \to H_2$ be ν-inverse strongly monotone mapping. Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be multi-valued maximal monotone mappings. Let Ω be a solution set of (8) - (9). Let $S : H_1 \to H_1$ be a κ-strictly pseudocontractive mapping and $F(S) \cap \Omega \neq \emptyset$. Let $\{x_n\}$ be the sequence generated for $x_0 \in H_1$ by*

$$\begin{cases} w_n = (1-\alpha_n)x_n, \\ y_n = J_\lambda^{B_1}(I-\lambda f_1)(w_n + \gamma A^*(J_\lambda^{B_2}(I-\lambda f_2) - I)Aw_n), \\ x_{n+1} = (1-\beta_n)y_n + \beta_n S y_n, \quad \forall n \geq 0, \end{cases} \tag{13}$$

where $0 < \lambda < 2\mu, 2\nu$ and $\gamma \in (0, \frac{1}{L})$, L is the spectral radius of the operator AA^* and A^* is the adjoint of A. Suppose $\{\alpha_n\}_{n=1}^\infty$ and $\{\beta_n\}_{n=1}^\infty$ are real sequences in $(0,1)$ satisfying the following conditions
(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$,
(ii) $0 < \liminf \beta_n \leq \limsup \beta_n < 1 - \kappa$,
then $\{x_n\}_{n=1}^\infty$ converges strongly to $p \in F(S) \cap \Omega$.

Recently, Deepho et. al [14] obtained the following result:

Theorem 2. *Let H_1 and H_2 be two real Hilbert spaces and let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed convex subsets. Let $A : H_1 \to H_2$ be a bounded linear linear operator. Let D be strongly positive bounded linear operator on H_1 with a coefficient $\bar{\tau} > 0$. Assume that $\{T_i\}_{i=1}^N : C \to H_1$ is a family of k_i-strictly pseudo-contraction mappings such that $\cap_{i=1}^N F(T_i) \cap \Omega_B \neq \emptyset$. Suppose that $f \in \Pi_C$ with a coefficient $\rho \in (0,1)$ and $\{\eta_i^{(n)}\}_{i=1}^N$ are finite sequences of positive numbers such that $\sum_{i=1}^N \eta_i^{(n)} = 1$ for all $n \geq 0$. For a given point $x_0 \in C$, $\alpha_n, \beta_n \in (0,1)$ and $0 < \tau < \frac{\bar{\tau}}{\rho}$, let $\{x_n\}$ be a sequence generated in the following:*

$$\begin{cases} u_n = J_\lambda^{B_1}(x_n + \gamma_n A^*(J_\lambda^{B_2} - I)Ax_n), \\ y_n = \beta_n u_n + (1-\beta_n)\sum_{i=1}^N \eta_{n,i} S_i u_n, \\ x_{n+1} = \alpha_n \tau G(x_n) + (I - \alpha_n D)y_n, \quad \forall n \geq 1, \end{cases} \tag{14}$$

where $\lambda > 0$ and $\gamma \in (0, \frac{1}{L})$, L is the spectral radius of the operator A^*A and A^* is the adjoint of A. Suppose the following conditions are satisfied:
(C1) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$ *and* $\sum_{n=1}^\infty |\alpha_n - \alpha_{n-1}| < \infty$,
(C2) $k_i \leq \beta_n \leq l < 1$, $\lim_{n\to\infty} \beta_n = l$ *and* $\sum_{n=1}^\infty |\beta_n - \beta_{n-1}| < \infty$,
(C3) $\sum_{n=1}^\infty \sum_{i=1}^N |\eta_i^{(n)} - \eta_i^{n-1}| < \infty$. *Then the sequence $\{x_n\}$ generated by (14) converges strongly to $q \in \cap_{i=1}^N F(T_i) \cap \Omega_B$ which solves the variational inequality*

$$\langle (D - \tau f)q, q - p \rangle \leq 0, \forall p \in \cap_{i=1}^N F(T_i) \cap \Omega_B.$$

In this paper, we present a general algorithm which does not require prior knowledge of the operator norm for solving split monotone variational inclusion problem, fixed point problem for a finite family of strictly pseudocontractive mappings and certain variational inequality problem. The result of this paper improve on the results of Shehu and Ogbuisi [28] and Deepho et. al [14] as follows:
1. The results of Shehu and Ogbuisi [28] and Deepho et. al [14] both require the knowledge of the operator norm while the result of this paper does not require any knowledge of the operator norm.
2. The result of Deepho et. al [14] took f_1 and f_2 to be identically zero but the result of this paper does not require f_1 and f_2 to be necessarily zero.
3. The result of this paper solve a variational inequality problem while the result of Shehu and Ogbuisi [28] did not do so.

2. Preliminaries

We start by stating some important results we will need in sequel.

Lemma 1. *[12,13] Let H be a Hilbert space and $T : H \to H$ a nonexpansive mapping, then for all $x, y \in H$,*

$$\langle (x - Tx) - (y - Ty), Ty - Tx \rangle \leq \frac{1}{2} \|(Tx - x) - (Ty - y)\|^2, \tag{15}$$

and consequently if $y \in F(T)$ then

$$\langle x - Tx, Ty - Tx \rangle \leq \frac{1}{2} \|Tx - x\|^2. \tag{16}$$

Lemma 2. *Let H be a real Hilbert space. Then the following result holds*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \forall\, x, y \in H.$$

Lemma 3. *Let H be a Hilbert space, then $\forall x, y \in H$ and $\alpha \in (0, 1)$, we have*

$$\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2.$$

Lemma 4. *(Demiclosedness principle) Let K be a nonempty, closed and convex subset of a real Hilbert space H. Let $T : K \to K$ be κ-strictly pseudocontractive mapping. Then $I - T$ is demi closed at 0, i.e., if $x_n \rightharpoonup x \in K$ and $x_n - Tx_n \to 0$, then $x = Tx$.*

Lemma 5. *[30] Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n \delta_n, \;\; n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in (0,1) and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that
(i) $\Sigma_{n=0}^{\infty} \gamma_n = \infty$
(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ or $\Sigma_{n=0}^{\infty} |\delta_n \gamma_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

Lemma 6. *[17] Let $M : H \to 2^H$ be a maximally monotone mapping and $f : H \to H$ be a Lipschitz continuous mapping. Then the mapping $G = M + f : H \to 2^H$ is a maximal monotone mapping.*

Lemma 7. *[22] Assume that D is a strongly positive linear operator on a Hilbert space with a coefficient $\delta > 0$ and $0 < \rho < ||D||^{-1}$. Then $||I - \rho D|| \leq 1 - \rho\delta$.*

Lemma 8. *[1] Let C be a nonempty closed convex subset of a Hilbert space H. Let $\{T_i\}_{i=1}^N : C \to H$ be a finite family of k_i- strictly pseudocontractive mappings and suppose $\{\eta_i\}_{i=1}^N$ is a positive sequence such that $\sum_{i=1}^N \eta_i = 1$. Then $\sum_{i=1}^N \eta_i T_i$ is a k- strictly pseudocontractive mapping with $k = \max\{k_i : 1 \le i \le N\}$.*

Lemma 9. *[1] Let C be a nonempty closed convex subset of a Hilbert space H. Let $\{T_i\}_{i=1}^N : C \to H$ be a finite family of k_i- strictly pseudocontractive mappings and suppose $\{\eta_i\}_{i=1}^N$ is a positive sequence such that $\sum_{i=1}^N \eta_i = 1$. Then $F(\sum_{i=1}^N \eta_i T_i) = \cap_{i=1}^N F(T_i)$.*

Lemma 10. *[1] Let C be a nonempty, closed and convex subset of a Hilbert space H. Assume that $f : C \to C$ is a contraction with a coefficient $\rho \in (0,1)$ and D is a strongly positive linear bounded operator with a coefficient $\bar{\delta} > 0$. Then, for $0 < \delta < \frac{\bar{\delta}}{\rho}$,*

$$\langle x - y, (D - \delta f)x - (D - \delta f)y \rangle \ge (\bar{\delta} - \delta\rho)||x - y||^2, \ \ \forall x, y \in H.$$

That is, $D - \delta f$ is strongly monotone with coefficient $\bar{\delta} - \delta\rho$.

A mapping $T : H \to H$ is said to be averaged if and only if it can be written as the average of the identity mapping and a nonexpansive mapping, i.e.,

$$T := (1 - \beta)I + \beta S$$

where $\beta \in (0,1)$ and $S : H \to H$ is a nonexpansive mapping and I is the identity mapping on H. Every averaged mapping is nonexpansive and every firmly nonexpansive mapping is averaged. Thus since the resolvent of maximal monotone operators are firmly nonexpansive, they are averaged and therefore nonexpansive. For details, please see [2,5,20,23].

3. Main Results

Theorem 3. *Let H_1 and H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator and A^* the adjoint of A. Let $f_1 : H_1 \to H_1$ be μ-inverse strongly monotone mapping and $f_2 : H_2 \to H_2$ be ν-inverse strongly monotone mapping. Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be multi-valued maximal monotone mappings. Let Ω be a solution set of (8) - (9), $S_i : H_1 \to H_1$ (i=1,2,...,N) be κ_i-strictly pseudocontractive mappings and $\mathbb{F} \cap \Omega \neq \emptyset$ where $\mathbb{F} = \cap_{i=1}^N F(S_i)$. Let D be a strongly positive bounded linear operator on H_1 with a coefficient $\bar{\delta} > 0$, G a ρ contraction on H_1, $0 < \delta < \dfrac{\bar{\delta}}{\rho}$ and $\{\eta_{n,i}\}_{i-1}^N \subset (0,1)$ are such that $\sum_{i=1}^N \eta_{n,i} = 1$. Let the step size γ_n be chosen in such a way that for some $\epsilon > 0$, $\gamma_n \in \Big(\epsilon, \dfrac{||(J_\lambda^{B_2}(I - \lambda f_2) - I)Aw_n||^2}{||A^*(J_\lambda^{B_2}(I - \lambda f_2) - I)Aw_n||^2} - \epsilon\Big)$ for $J_\lambda^{B_2}(I - \lambda f_2)Aw_n \neq Aw_n$ and $\gamma_n = \gamma$, otherwise (γ being any nonnegative real number). Then the sequences $\{w_n\}, \{x_n\}$ and $\{y_n\}$ generated iteratively for an arbitrary $x_0 \in C$ and a fixed $u \in C$ by*

$$\begin{cases} w_n = (I - \alpha_n D)x_n + \alpha_n \delta G(x_n), \\ y_n = J_\lambda^{B_1}(I - \lambda f_1)(w_n + \gamma_n A^*(J_\lambda^{B_2}(I - \lambda f_2) - I)Aw_n), \\ x_{n+1} = (1 - \beta_n)y_n + \beta_n \sum_{i=1}^N \eta_{n,i} S_i y_n, \ \ \forall n \ge 0, \end{cases} \tag{17}$$

converges strongly to a point $p \in \Omega \cap \mathbb{F}$ which is also a solution of the variational inequality

$$\langle (D - \delta G)p, p - q \rangle \le 0, \ \ \forall q \in \Omega \cap \mathbb{F},$$

where $\lambda > 0$ is such that where $0 < \lambda < 2\mu, 2\nu$ and $\{\alpha_n\}_{n=1}^\infty$ and $\{\beta_n\}_{n=1}^\infty$ are real sequences in $(0,1)$ satisfying the following conditions
(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$,
(ii) $\kappa = \max\{\kappa_i : 1 \le i \le N\}$, $0 < \liminf \beta_n \le \limsup \beta_n < 1 - \kappa$.

Proof. For any $x, y \in H$, we have

$$\begin{aligned}
||P_{\Omega\cap\mathbb{F}}(I-D+\delta G)(x)-P_{\Omega\cap\mathbb{F}}(I-D+\delta G)(y)| &\leq ||(I-D+\delta G)(x)-(I-D+\delta G)(y)|| \\
&\leq ||(I-D)x-(I-D)y||+\delta||Gx-Gy|| \\
&\leq (1-\bar{\delta})||x-y||+\delta\rho||x-y|| \\
&\leq (1-(\bar{\delta}-\delta\rho)||x-y||.
\end{aligned}$$

Thus $P_{\Omega\cap\mathbb{F}}(I-D+\delta G)$ is a contraction and by the Banach contraction mapping principle, we conclude that there exists $p \in H$ such that $p = P_{\Omega\cap\mathbb{F}}(I-D+\delta G)p$.
Next we show that $\{x_n\}$ is bounded.

$$\begin{aligned}
||w_n-p|| &= ||(I-\alpha_n D)(x_n-p)+\alpha_n(\delta G(x_n)-Dp)|| \\
&\leq (1-\alpha_n\bar{\delta})||x_n-p||+\alpha_n||\delta G(x_n)-Dp|| \\
&\leq (1-\alpha_n\bar{\delta})||x_n-p||+\alpha_n||\delta G(x_n)-\delta G(p)||+\alpha_n||\delta G(p)-Dp)|| \\
&\leq [1-(\bar{\delta}-\delta\rho)\alpha_n]||x_n-p||+\alpha_n||\delta G(p)-Dp)||. \qquad (18)
\end{aligned}$$

But

$$\begin{aligned}
||y_n-p||^2 &= ||J_\lambda^{B_1}(I-\lambda f_1)(w_n+\gamma_n A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n)-p||^2 \\
&\leq ||w_n+\gamma_n A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n-p||^2 \\
&= ||w_n-p||^2+\gamma_n^2||A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2 \\
&\quad +2\gamma_n\langle w_n-p, A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n\rangle. \qquad (19)
\end{aligned}$$

Now by Lemma 1 (16), we have

$$\begin{aligned}
2\gamma_n\langle w_n-p, A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n\rangle &= 2\gamma_n\langle A(w_n-p),(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n\rangle \\
&= 2\gamma_n[\langle J_\lambda^{B_2}(I-\lambda f_2)Aw_n-Ap,(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n\rangle \\
&\quad -||(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2] \\
&\leq 2\gamma_n[\frac{1}{2}||(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2 \\
&\quad -||(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2] \\
&= -\gamma_n||(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2. \qquad (20)
\end{aligned}$$

Thus from (19) and (20), we have

$$\begin{aligned}
||y_n-p||^2 &\leq ||w_n-p||^2+\gamma_n^2||A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2-\gamma_n||(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2 \\
&= ||w_n-p||^2+\gamma_n[\gamma_n||A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2 \\
&\quad -||(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2]. \qquad (21)
\end{aligned}$$

Hence from the condition $\gamma_n \in \Big(\epsilon, \dfrac{||(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2}{||A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2}-\epsilon\Big)$, we obtain

$$||y_n-p||^2 \leq ||w_n-p||^2. \qquad (22)$$

$$
\begin{aligned}
||x_{n+1}-p||^2 &= ||(1-\beta_n)y_n+\beta_n\sum_{i=1}^{N}\eta_{n,i}S_iy_n-p||^2||^2\\
&= (1-\beta_n)||y_n-p||^2+\beta_n||\sum_{i=1}^{N}\eta_{n,i}S_iy_n-p||^2-\beta_n(1-\beta_n)||y_n-\sum_{i=1}^{N}\eta_{n,i}S_iy_n||^2\\
&\leq (1-\beta_n)||y_n-p||^2+\beta_n[||y_n-p||^2+\kappa||y_n-\sum_{i=1}^{N}\eta_{n,i}S_iy_n||^2]\\
&\quad -\beta_n(1-\beta_n)||y_n-\sum_{i=1}^{N}\eta_{n,i}S_iy_n||^2\\
&= ||y_n-p||^2-\beta_n(1-\beta_n-\kappa)||y_n-\sum_{i=1}^{N}\eta_{n,i}S_iy_n||^2\\
&\leq ||y_n-p||^2. \qquad (23)
\end{aligned}
$$

Therefore, from (18), we have

$$
\begin{aligned}
||x_{n+1}-p|| &\leq ||w_n-p||\\
&\leq [1-(\bar{\delta}-\delta\rho)\alpha_n]||x_n-p||+\alpha_n||\delta G(p)-Dp)||\\
&= [1-(\bar{\delta}-\delta\rho)\alpha_n]||x_n-p||+(\bar{\delta}-\delta\rho)\alpha_n\frac{1}{(\bar{\delta}-\delta\rho)}||\delta G(p)-Dp)||\\
&\leq \max\{||x_n-p||,\frac{1}{(\bar{\delta}-\delta\rho)}||\delta G(p)-Dp)||\}\\
&\vdots\\
&\leq \max\{||x_0-p||,\frac{1}{(\bar{\delta}-\delta\rho)}||\delta G(p)-Dp)||\}. \qquad (24)
\end{aligned}
$$

We then conclude that $\{x_n\}$ is bounded.
Again,

$$
\begin{aligned}
||w_n-x_n|| &= ||(I-\alpha_nD)x_n+\alpha_n\delta G(x_n)-x_n||\\
&= \alpha_n||Dx_n-\delta G(x_n)||\to 0, n\to\infty. \qquad (25)
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
||x_{n+1}-p||^2 &\leq ||y_n-p||^2-\beta_n(1-\beta_n-\kappa)||y_n-\sum_{i=1}^{N}\eta_{n,i}S_iy_n||^2\\
&\leq ||w_n-p||^2-\beta_n(1-\beta_n-\kappa)||y_n-\sum_{i=1}^{N}\eta_{n,i}S_iy_n||^2\\
&= ||(I-\alpha_nD)x_n+\alpha_n\delta G(x_n)-p||^2-\beta_n(1-\beta_n-\kappa)||y_n-\sum_{i=1}^{N}\eta_{n,i}S_iy_n||^2\\
&\leq ||x_n-p||^2+\alpha_n^2||Dx_n-\delta G(x_n)||^2-2\alpha_n\langle x_n-p,Dx_n-\delta G(x_n)\rangle\\
&\quad -\beta_n(1-\beta_n-\kappa)||y_n-\sum_{i=1}^{N}\eta_{n,i}S_iy_n||^2. \qquad (26)
\end{aligned}
$$

We divide into two cases to obtain strong convergence.
Case 1. Assume that $\{||x_n-p||^2\}$ is a monotonically decreasing sequence. It then follows that $\{||x_n-p||^2\}$ is convergent and

$$
||x_n-p||-||x_{n+1}-p||\to 0, n\to\infty. \qquad (27)
$$

Therefore, from (26), we have

$$\begin{aligned}\beta_n(1-\beta_n-\kappa)||y_n-\sum_{i=1}^{N}\eta_{n,i}S_iy_n||^2 &\leq ||x_n-p||^2-||x_{n+1}-p||^2+\alpha_n^2||Dx_n-\delta G(x_n)||^2\\ &\quad -2\alpha_n\langle x_n-p,Dx_n-\delta G(x_n)\rangle\to 0, n\to\infty.\end{aligned}$$

That is,

$$||y_n-\sum_{i=1}^{N}\eta_{n,i}S_iy_n||\to 0, n\to\infty. \tag{28}$$

Also,

$$\begin{aligned}||x_{n+1}-p||^2 &\leq ||y_n-p||^2\\ &\leq ||w_n-p||^2+\gamma_n[\gamma_n||A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2-||(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2]\\ &\leq ||x_n-p||^2+\alpha_n^2||Dx_n-\delta G(x_n)||^2-2\alpha_n\langle x_n-p,Dx_n-\delta G(x_n)\rangle\\ &\quad +\gamma_n[\gamma_n||A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2-||(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2].\end{aligned} \tag{29}$$

It then follows from (29) and the condition

$$\gamma_n\in\left(\epsilon,\frac{||(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2}{||A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2}-\epsilon\right),$$

that

$$\begin{aligned}||x_{n+1}-p||^2 &\leq ||x_n-p||^2+\alpha_n^2||Dx_n-\delta G(x_n)||^2-2\alpha_n\langle x_n-p,Dx_n-\delta G(x_n)\rangle\\ &\quad -\epsilon^2||A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2,\end{aligned} \tag{30}$$

which implies

$$\begin{aligned}\epsilon^2||A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2 &\leq ||x_n-p||^2-||x_{n+1}-p||^2+\alpha_n^2||Dx_n-\delta G(x_n)||^2\\ &\quad -2\alpha_n\langle x_n-p,Dx_n-\delta G(x_n)\rangle\to 0, n\to\infty.\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}||A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||=0. \tag{31}$$

Also from (29), we have

$$\begin{aligned}\gamma_n||(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2 &\leq ||x_n-p||^2-||x_{n+1}-p||^2+\alpha_n^2||Dx_n-\delta G(x_n)||^2\\ &\quad -2\alpha_n\langle x_n-p,Dx_n-\delta G(x_n)\rangle\\ &\quad +\gamma_n^2||A^*(J_\lambda^{B_2}(I-\lambda f_2)-I)Aw_n||^2\to 0, n\to\infty.\end{aligned} \tag{32}$$

Further,

$$\begin{aligned}
||y_n - p||^2 &= ||J_\lambda^{B_1}(I-\lambda f_1)(w_n + \gamma_n A^*(J_\lambda^{B_2}(I-\lambda f_2) - I)Aw_n) - p||^2 \\
&\leq \langle y_n - p, w_n + \gamma_n A^*(J_\lambda^{B_2}(I-\lambda f_2) - I)Aw_n - p\rangle \\
&= \frac{1}{2}[||y_n - p||^2 + ||w_n + \gamma_n A^*(J_\lambda^{B_2}(I-\lambda f_2) - I)Aw_n - p||^2 \\
&\quad -||y_n - p - (w_n + \gamma_n A^*(J_\lambda^{B_2}(I-\lambda f_2) - I)Aw_n) - p)||^2] \\
&\leq \frac{1}{2}[||y_n - p||^2 + ||w_n - p||^2 \\
&\quad +\gamma_n(\gamma_n||A^*(J_\lambda^{B_2}(I-\lambda f_2) - I)Aw_n||^2 - ||(J_\lambda^{B_2}(I-\lambda f_2) - I)Aw_n||^2) \\
&\quad -||y_n - p - (w_n + \gamma_n A^*(J_\lambda^{B_2}(I-\lambda f_2) - I)Aw_n - p)||^2] \\
&\leq \frac{1}{2}[||y_n - p||^2 + ||w_n - p||^2 - (||y_n - w_n||^2 + \gamma_n^2||A^*(J_\lambda^{B_2}(I-\lambda f_2) - I)Aw_n||^2 \\
&\quad -2\gamma_n\langle y_n - w_n, A^*(J_\lambda^{B_2}(I-\lambda f_2) - I)Aw_n\rangle)] \\
&\leq \frac{1}{2}[||y_n - p||^2 + ||w_n - p||^2 - ||y_n - w_n||^2 \\
&\quad +2\gamma_n||y_n - w_n||||A^*(J_\lambda^{B_2}(I-\lambda f_2) - I)Aw_n||].
\end{aligned} \tag{33}$$

That is,

$$\begin{aligned}
||y_n - p||^2 &\leq ||w_n - p||^2 - ||y_n - w_n||^2 \\
&\quad +2\gamma_n||y_n - w_n||||A^*(J_\lambda^{B_2}(I-\lambda f_2) - I)Aw_n||.
\end{aligned} \tag{34}$$

Thus, it follows from (23) and (34) that

$$\begin{aligned}
||x_{n+1} - p||^2 &\leq ||w_n - p||^2 - ||y_n - w_n||^2 \\
&\quad +2\gamma_n||y_n - w_n||||A^*(J_\lambda^{B_2}(I-\lambda f_2) - I)Aw_n||.
\end{aligned} \tag{35}$$

Hence,

$$\begin{aligned}
||y_n - w_n||^2 &\leq ||w_n - p||^2 - ||x_{n+1} - p||^2 + 2\gamma_n||y_n - w_n||||A^*(J_\lambda^{B_2}(I-\lambda f_2) - I)Aw_n|| \\
&\leq ||x_n - p||^2 + \alpha_n^2||Dx_n - \delta G(x_n)||^2 - 2\alpha_n\langle x_n - p, Dx_n - \delta G(x_n)\rangle \\
&\quad -||x_{n+1} - p||^2 + 2\gamma_n||y_n - w_n||||A^*(J_\lambda^{B_2}(I-\lambda f_2) - I)Aw_n|| \to 0, n \to \infty.
\end{aligned} \tag{36}$$

Furthermore,

$$||x_n - y_n|| \leq ||x_n - w_n|| + ||w_n - y_n|| \to 0, n \to \infty. \tag{37}$$

$$||x_{n+1} - y_n|| = \beta_n||y_n - \sum_{i=1}^{N}\eta_{n,i}S_i y_n|| \to 0, n \to \infty, \tag{38}$$

and hence,

$$||x_{n+1} - x_n|| \leq ||x_{n+1} - y_n|| + ||x_n - y_n|| \to 0, \to \infty. \tag{39}$$

Let $u_n = w_n + \gamma A^*(J_\lambda^{B_2}(I-\lambda f_2) - I)Aw_n$, then

$$\|u_n - w_n\|^2 = L\gamma^2\|(J_\lambda^{B_2}(I-\lambda f_2) - I)Aw_n\|^2 \to 0. \tag{40}$$

Combining (36) and (40), we have that

$$\|y_n - u_n\| \leq \|y_n - w_n\| + \|w_n - u_n\| \to 0. \tag{41}$$

Next, we show that

$$\limsup_{n\to\infty} \langle (D - \delta G)q, q - x_n \rangle \leq 0.$$

We choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\lim_{i\to\infty} \langle (D - \delta G)q, q - x_{n_i} \rangle = \limsup_{n\to\infty} \langle (D - \delta G)q, q - x_n \rangle.$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence of $\{x_{n_i}\}$ also denoted as $\{x_{n_i}\}$ that converges weakly to some $q \in H$ and consequently we have $\{y_{n_i}\}$ and $\{w_{n_i}\}$ converge weakly to q. From Lemma 4 , Lemma 8, Lemma 9 and (28), we conclude $q \in \mathbb{F}$.

We now show that $q \in I(f_1, B_1)$. Since f_1 is a $\frac{1}{\mu}$-Lipschitz monotone mapping and the domain of f_1 is H_1 then by Lemma 6 we conclude that $B_1 + f_1$ is maximally monotone. Let $(v, z) \in G(B_1 + f_1)$, that is $z - f_1 v \in B_1(v)$.

Since $y_{n_i} = J_\lambda^{B_1}(I - \lambda f_1)u_{n_i}$ we obtain that

$$(I - \lambda f_1)u_{n_i} \in (I + \lambda B_1)y_{n_i}.$$

That is,

$$\frac{1}{\lambda}(u_{n_i} - \lambda f_1 u_{n_i} - y_{n_i}) \in B_1(y_{n_i}).$$

Using the maximal monotonicity of $(B_1 + f_1)$, we have

$$\langle v - y_{n_i}, z - f_1 v - \frac{1}{\lambda}(u_{n_i} - \lambda f_1 u_{n_i} - y_{n_i}) \rangle \geq 0.$$

Therefore,

$$\begin{aligned} \langle v - y_{n_i}, z \rangle &\geq \langle v - y_{n_i}, f_1 v + \frac{1}{\lambda}(u_{n_i} - \lambda f_1 u_{n_i} - y_{n_i}) \rangle \\ &= \langle v - y_{n_i}, f_1 v - f_1 y_{n_i} + f_1 y_{n_i} - f_1 u_{n_i} + \frac{1}{\lambda}(u_{n_i} - y_{n_i}) \rangle \\ &\geq 0 + \langle v - y_{n_i}, f_1 y_{n_i} - f_1 u_{n_i} \rangle + \langle v - y_{n_i}, \frac{1}{\lambda}(u_{n_i} - y_{n_i}) \rangle. \end{aligned} \tag{42}$$

By (41), we obtain that

$$\lim_{i\to\infty} \|f_1 y_{n_i} - f_1 u_{n_i}\| = 0.$$

Also, since $y_{n_i} \rightharpoonup q$, we have

$$\lim_{i\to\infty} \langle v - y_{n_i}, z \rangle = \langle v - p, z \rangle.$$

Thus, from (42)

$$\langle v - q, z \rangle \geq 0.$$

Since $B_1 + f_1$ is maximally monotone, we have $0 \in (B_1 + f_1)q$ which implies that

$$q \in I(f_1, B_1).$$

Moreover, we have Aw_{n_i} converges weakly to Aq, thus by (32) and the fact that $J_\lambda^{B_2}(I-\lambda f_2)$ is nonexpansive, then by Lemma 4, we get that

$$0 \in f_2 Aq + B_2(Aq).$$

That is $Aq \in I(f_2, B_2)$. Hence, $q \in \Omega \cap \mathbb{F}$.
Since $p = P_{\Omega\cap\mathbb{F}}(I - D + \delta G)p$ and $q \in \Omega \cap \mathbb{F}$, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle (D-\delta G)p, p - x_n\rangle &= \lim_{i\to\infty}\langle (D-\delta G)p, p-x_{n_i}\rangle \\ &= \langle (D-\delta G)p, p-q\rangle \le 0. \end{aligned} \tag{43}$$

We now show that $\{x_n\}$ converges strongly to p.

$$\begin{aligned} ||x_{n+1}-P||^2 &\le ||y_n - p||^2 \\ &\le ||w_n - p||^2 \\ &= ||(I-\alpha_n D)x_n + \alpha_n\delta G(x_n) - p||^2 \\ &\le (1-\alpha_n\bar{\delta})^2||x_n-p||^2 + \alpha_n^2||\delta G(x_n) - Dp||^2 \\ &\quad +2\alpha_n\langle (I-\alpha_n D)(x_n-p), \delta G(x_n) - Dp\rangle \\ &\le (1-\alpha_n\bar{\delta})^2||x_n-p||^2 + \alpha_n^2||\delta G(x_n) - Dp||^2 \\ &\quad +2\alpha_n\langle x_n - p, \delta G(x_n) - Dp\rangle - 2\alpha_n^2\langle Dx_n - Dp, \delta G(x_n) - Dp\rangle \\ &\le (1-\alpha_n\bar{\delta})^2||x_n-p||^2 + \alpha_n^2||\delta G(x_n) - Dp||^2 + 2\alpha_n\delta\langle x_n - p, G(x_n) - G(p)\rangle \\ &\quad +2\alpha_n\langle x_n - p, \delta G(p) - Dp\rangle - 2\alpha_n^2\langle Dx_n - Dp, \delta G(x_n) - Dp\rangle \\ &\le (1-\alpha_n\bar{\delta})^2||x_n-p||^2 + \alpha_n^2||\delta G(x_n) - Dp||^2 + 2\alpha_n\rho\delta||x_n-p||^2 \\ &\quad +2\alpha_n\langle x_n - p, \delta G(p) - Dp\rangle - 2\alpha_n^2\langle Dx_n - Dp, \delta G(x_n) - Dp\rangle \\ &= (1-2\alpha_n(\bar{\delta}-\rho\delta) + \alpha_n^2\bar{\delta}^2)||x_n-p||^2 + \alpha_n^2||\delta G(x_n) - Dp||^2 \\ &\quad +2\alpha_n\langle x_n - p, \delta G(p) - Dp\rangle - 2\alpha_n^2\langle Dx_n - Dp, \delta G(x_n) - Dp\rangle. \end{aligned}$$

Therefore,

$$\begin{aligned} ||x_{n+1}-P||^2 \le\ & (1-\alpha_n(\bar{\delta}-\rho\delta))||x_n-p||^2 + \alpha_n(\bar{\delta}-\rho\delta)[\frac{1}{(\bar{\delta}-\rho\delta)}(\alpha_n\bar{\delta}^2||x_n-p||^2 \\ & +\alpha_n||\delta G(x_n) - Dp||^2 + 2\langle x_n - p, \delta G(p) - Dp\rangle - 2\alpha_n\langle Dx_n - Dp, \delta G(x_n) - Dp\rangle)]. \end{aligned}$$

Thus by Lemma 5, we have $x_n \to p$.
Case 2. Assume that $\{\|x_n - p\|\}$ is not a monotonically decreasing sequence. Set $\Gamma_n = \|x_n - p\|^2$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \ge n_0$ (for some n_0 large enough) defined by

$$\tau(n) := \max\{k \in \mathbb{N} : k \ge n, \Gamma_k \le \Gamma_{k+1}\}.$$

Clearly τ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and $\Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1}$, for $n \ge n_0$. Then from (26), we have

$$\begin{aligned} 0 &\le ||x_{\tau(n)+1} - p||^2 - ||x_{\tau(n)} - p||^2 \\ &\le \alpha_{\tau(n)}^2||Dx_{\tau(n)} - \delta G(x_{\tau(n)})||^2 - 2\alpha_{\tau(n)}\langle x_{\tau(n)} - p, Dx_{\tau(n)} - \delta G(x_{\tau(n)})\rangle \\ &\quad -\beta_{\tau(n)}(1-\beta_{\tau(n)}-\kappa)||y_{\tau(n)} - \sum_{i=1}^N \eta_{\tau(n),i}S_i y_{\tau(n)}||^2, \end{aligned}$$

which implies

$$\begin{aligned}\beta_{\tau(n)}(1-\beta_{\tau(n)}-\kappa)||y_{\tau(n)}-\sum_{i=1}^{N}\eta_{\tau(n),i}S_iy_{\tau(n)}||^2 &\leq \alpha^2_{\tau(n)}||Dx_{\tau(n)}-\delta G(x_{\tau(n)})||^2\\ &-2\alpha_{\tau(n)}\langle x_{\tau(n)}-p, Dx_{\tau(n)}-\delta G(x_{\tau(n)})\rangle \to 0.\end{aligned}$$

By the same argument as in (28)-(43) in case 1, we conclude that

$$\limsup_{n\to\infty}\langle (D-\delta G)q, q-x_{\tau(n)}\rangle \leq 0.$$

Hence, for all $n \geq n_0$,

$$\begin{aligned}0 &\leq ||x_{\tau(n)+1}-p||^2-||x_{\tau(n)}-p||^2\\ &\leq (1-\alpha_{\tau(n)}(\bar{\delta}-\rho\delta))||x_{\tau(n)}-p||^2\\ &+\alpha_{\tau(n)}(\bar{\delta}-\rho\delta)[\frac{1}{(\bar{\delta}-\rho\delta)}(\alpha_{\tau(n)}\bar{\delta}^2||x_{\tau(n)}-p||^2+\alpha_{\tau(n)}||\delta G(x_{\tau(n)})-Dp||^2\\ &+2\langle x_{\tau(n)}-p, \delta G(p)-Dp\rangle-2\alpha_{\tau(n)}\langle Dx_{\tau(n)}-Dp, \delta G(x_{\tau(n)})-Dp\rangle)]-||x_{\tau(n)}-p||^2\\ &= \alpha_{\tau(n)}(\bar{\delta}-\rho\delta)[\frac{1}{(\bar{\delta}-\rho\delta)}(\alpha_{\tau(n)}\bar{\delta}^2||x_{\tau(n)}-p||^2+\alpha_{\tau(n)}||\delta G(x_{\tau(n)})-Dp||^2\\ &+2\langle x_{\tau(n)}-p, \delta G(p)-Dp\rangle-2\alpha_{\tau(n)}\langle Dx_{\tau(n)}-Dp, \delta G(x_{\tau(n)})-Dp\rangle)-||x_{\tau(n)}-p||^2].\end{aligned}$$

That is,

$$\begin{aligned}||x_{\tau(n)}-p||^2 &\leq \frac{1}{(\bar{\delta}-\rho\delta)}(\alpha_{\tau(n)}\bar{\delta}^2||x_{\tau(n)}-p||^2+\alpha_{\tau(n)}||\delta G(x_{\tau(n)})-Dp||^2\\ &+2\langle x_{\tau(n)}-p, \delta G(p)-Dp\rangle\\ &-2\alpha_{\tau(n)}\langle Dx_{\tau(n)}-Dp, \delta G(x_{\tau(n)})-Dp\rangle)\to 0, n\to\infty.\end{aligned}\tag{44}$$

Therefore,

$$\|x_{\tau(n)}-p\|^2 \leq \alpha_{\tau(n)}\|p\|^2-2\alpha_{\tau(n)}(1-\alpha_{\tau(n)})\langle x_{\tau(n)}-p, p\rangle \to 0.$$

Thus,

$$\lim_{n\to\infty}\|x_{\tau(n)}-p\|^2=0,$$

and hence,

$$\lim_{n\to\infty}\Gamma_{\tau(n)}=\lim_{n\to\infty}\Gamma_{\tau(n)+1}.$$

Furthermore, for $n \geq n_0$, it is observed that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$(that is $\tau(n) < n$) because $\Gamma_j > \Gamma_{j+1}$ for $\tau(n)+1 \leq j \leq n$. Consequently for all $n \geq n_0$,

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)}+1\} = \Gamma_{\tau(n)}+1.$$

So $\lim_{n\to\infty}\Gamma_n = 0$, that is $\{x_n\}$,$\{y_n\}$ and $\{w_n\}$ converge strongly to $p \in \Omega\cap\mathbb{F}$ which is also a solution of the variational inequality

$$\langle (D-\delta G)p, p-q\rangle \leq 0, \;\; \forall q \in \Omega\cap\mathbb{F}.$$

Corollary 1. *Let H_1 and H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator and A^* the adjoint of A. Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be multi-valued maximal monotone mappings. Let $S_i : H_1 \to H_1$ (i=1,2,...,N) be κ_i-strictly pseudocontractive mappings and $\mathbb{F}\cap\Omega_B \neq \emptyset$ where $\mathbb{F} = \cap_{i=1}^{N}F(S_i)$.*

Let D be a strongly positive bounded linear operator on H_1 with a coefficient $\bar{\delta} > 0$, G a ρ contraction on H_1, $0 < \delta < \dfrac{\bar{\delta}}{\rho}$ and $\{\eta_{n,i}\}_{i-1}^{N} \subset (0,1)$ are such that $\sum_{i=1}^{N} \eta_{n,i} = 1$. Let the step size γ_n be chosen in such a way that for some $\epsilon > 0$, $\gamma_n \in \Big(\epsilon, \dfrac{||(J_\lambda^{B_2} - I)Aw_n||^2}{||A^*(J_\lambda^{B_2} - I)Aw_n||^2} - \epsilon\Big)$ for $J_\lambda^{B_2} Aw_n \neq Aw_n$ and $\gamma_n = \gamma$, otherwise (γ being any nonnegative real number). Then the sequences $\{w_n\}, \{x_n\}$ and $\{y_n\}$ generated iteratively for an arbitrary $x_0 \in C$ and a fixed $u \in C$ by

$$\begin{cases} w_n = (I - \alpha_n D)x_n + \alpha_n \delta G(x_n), \\ y_n = J_\lambda^{B_1}(w_n + \gamma_n A^*(J_\lambda^{B_2} - I)Aw_n), \\ x_{n+1} = (1-\beta_n)y_n + \beta_n \sum_{i=1}^{N} \eta_{n,i} S_i y_n, \ \ \forall n \geq 0, \end{cases} \tag{45}$$

converges strongly to a point $p \in \Omega \cap \mathbb{F}$ which is also a solution of the variational inequality

$$\langle (D - \delta G)p, p - q \rangle \leq 0, \ \ \forall q \in \Omega_B \cap \mathbb{F},$$

where $\lambda > 0$ is a positive real number and $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are real sequences in $(0,1)$ satisfying the following conditions
(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty,$
(ii) $\kappa = \max\{\kappa_i : 1 \leq i \leq N\}, \ 0 < \liminf \beta_n \leq \limsup \beta_n < 1 - \kappa.$

Corollary 2. *Let H_1 and H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator and A^* the adjoint of A. Let $f_1 : H_1 \to H_1$ be μ-inverse strongly monotone mapping and $f_2 : H_2 \to H_2$ be ν-inverse strongly monotone mapping. Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be multi-valued maximal monotone mappings. Let Ω be a solution set of (8) - (9), $S_i : H_1 \to H_1$ (i=1,2,...,N) be nonexpansive mappings and $\mathbb{F} \cap \Omega \neq \emptyset$ where $\mathbb{F} = \cap_{i=1}^{N} F(S_i)$. Let D be a strongly positive bounded linear operator on H_1 with a coefficient $\bar{\delta} > 0$, G a ρ contraction on H_1, $0 < \delta < \dfrac{\bar{\delta}}{\rho}$ and $\{\eta_{n,i}\}_{i-1}^{N} \subset (0,1)$ are such that $\sum_{i=1}^{N} \eta_{n,i} = 1$. Let the step size γ_n be chosen in such a way that for some $\epsilon > 0$, $\gamma_n \in \Big(\epsilon, \dfrac{||(J_\lambda^{B_2}(I - \lambda f_2) - I)Aw_n||^2}{||A^*(J_\lambda^{B_2}(I - \lambda f_2) - I)Aw_n||^2} - \epsilon\Big)$ for $J_\lambda^{B_2}(I - \lambda f_2)Aw_n \neq Aw_n$ and $\gamma_n = \gamma$, otherwise (γ being any nonnegative real number). Then the sequences $\{w_n\}, \{x_n\}$ and $\{y_n\}$ generated iteratively for an arbitrary $x_0 \in C$ and a fixed $u \in C$ by*

$$\begin{cases} w_n = (I - \alpha_n D)x_n + \alpha_n \delta G(x_n), \\ y_n = J_\lambda^{B_1}(I - \lambda f_1)(w_n + \gamma_n A^*(J_\lambda^{B_2}(I - \lambda f_2) - I)Aw_n), \\ x_{n+1} = (1-\beta_n)y_n + \beta_n \sum_{i=1}^{N} \eta_{n,i} S_i y_n, \ \ \forall n \geq 0, \end{cases} \tag{46}$$

converges strongly to a point $p \in \Omega \cap \mathbb{F}$ which is also a solution of the variational inequality

$$\langle (D - \delta G)p, p - q \rangle \leq 0, \ \ \forall q \in \Omega \cap \mathbb{F},$$

where $\lambda > 0$ is such that where $0 < \lambda < 2\mu, 2\nu$ and $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are real sequences in $(0,1)$ satisfying the following conditions
(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty,$
(ii) $0 < \liminf \beta_n \leq \limsup \beta_n < 1.$

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