Spectrally negative stable vectors, their covariations on the positive orthant, and the Capital Asset Pricing Model

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Spectrally negative stable vectors, their covariations on the positive orthant, and the Capital Asset Pricing Model

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Keywords: Capital Asset Pricing Model (CAPM); stochastic dominance; pseudo-isotropic distributions; stable distributions; spectrally negative distributions; covariation

MSC: 91G19, 60Exx, 60H05, 49K45.

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1. Introduction

Among asset market models, *"the grandfather of them all, the celebrated Capital Asset Pricing Model"* (quoting Varian's textbook [1, p. 371]) has become ubiquitous over its five decades. For a brief historical account of its (pre-) history from Treynor, Sharpe, Lintner and Mossin, see Sullivan [2]. There is a vast empirical literature wherein its performance has been tested and criticized, the tests and test statistics have been scrutinized, and its mere testability has been discussed over; empirical performance is *not* the topic of this paper, but see, e.g., Jagannathan and McGrattan [3] or Johnstone [4].

The CAPM has shown up in several variants, but in the common textbook version, it will enjoy *inter alia* the following properties, which are key points to this paper:

- The possible combinations of mean and dispersion (as measured by standard deviation) attainable in portfolios with zero position in the risk-free opportunity, form a strictly convex set (the *Markowitz bullet*) in the (stdev,mean) plane.
- Allowing to use the risk-free opportunity as well, the mean–variance efficient returns are characterized by an increasing line from the risk-free return on the second axis, tangent to the Markowitz bullet at a single unique point (the *market portfolio:* the aggregate risky portfolio must be a scaling of this, as every agent's risky portfolio is).
- Efficiency of the market portfolio determines prices up to one degree of freedom – the pricing of the market portfolio itself, reflecting the market price of risk.

- Once the market portfolio's return is given, other prices can be characterized in terms of the coefficient *beta*, the ratio between the opportunity's expected return in excess of risk-free, and the corresponding one of the market portfolio. In the standard version, the beta is computed as the opportunity's covariance with the market, divided by the market portfolio's variance. Equivalently, it is the relative change in standard deviation transferring one (infinitesimal) monetary unit from the market portfolio to the opportunity.

Key to the CAPM is the monetary two-fund separation theorem, a key result in modern portfolio theory since Tobin [5]: the property that all agents could be satisfied by the risk-free opportunity and one single risky portfolio (namely, the market portfolio in CAPM) and forming a linear combination like above. Most prominently, the property holds when mean–variance trade-off is well-justified (see Johnstone and Lindley [6] and the references therein for historical accounts); that is, the elliptical distributions class (Owen and Rabinovitch [7] and Chamberlain [8]); let us therfore refer to this classical framework as *elliptical CAPM*, although, arguably, the multinormal special case is the more common assumption. Let us briefly discuss some extensions which are relevant to this paper:

- The risky returns vector could be shifted symmetric stable, if integrable. This is a consequence of the monetary two-fund separation property, for which early cases were given by Fama [9] and Samuelson [10]; Fama [11] refers to an earlier unpublished 1967 note of his on the stable CAPM. Further CAPMs for (shifted) symmetric stable returns are given by Belkacem et al. [12] and Gamrowski and Rachev [13], and indeed, symmetric stability can be generalized to so-called *pseudo-isotropic* distributions (this author [14]), weakening integrability.
- Elliptical – and stable/pseudo-isotropic – CAPM admits the "no short sale" restriction on the risky investment opportunities (with the obvious reservation that only investment opportunities wich are taken on, will be completely priced, while we only know about those left untouched that they are too expensive to justify its return). Ross [15] points out that all that is needed, is the monetary two-fund separation property.
- If there is no risk-free opportunity, the *minimum-variance portfolio* can substitute for the risk-free in the elliptical portfolio separtion theorem. Thus, elliptical CAPM can be established without any risk-free opportunity, the so-called Black [16] zero-beta CAPM.
- However, the absence of risk-free opportunity can not be combined with the absence of short sale opportunities ([15]). And neither can it be combined with shifted symmetric risky returns, except very special cases; Fama [11, section VI.B] claims the risk-free opportunity «greatly simplifies determination of the efficient set of portfolios», and indeed it turns out ([14, Theorem 11]) – that without the risk-free opportunity, the efficient portfolios no longer form a convex set.

On the other hand, the absence of short sale opens new possibilities: much like how the restriction to elliptical (or shifted pseudo-isotropic) distributions restricts the *attainable portfolio returns distributions*, so does the restriction to nonnegative investments. For example, if all returns distributions are iid, stable and integrable, then the *convex* combinations will only differ in their scale parameter (which generalizes standard deviation). Except under (shifted) symmetry, this property breaks down once negative coefficients are allowed. The returns distributions considered in this paper will not admit the CAPM over unrestricted portfolios. For the spectrally negative stable class, the two-fund monetary separation properties are proven in [17, Section 5] (showing that continuous-time properties are inherited from a single-period model); this paper gives conditions for a CAPM with, in particular, a *beta* for the equilibrium prices, the uniqueness of such one, and the covariation measure of comovement is extended to the distribution class.

In addition to the spectrally negative stable class, the paper introduces an apparently new class of multivariate distributions for which the CAPM construction goes through, and proposes a generalization of covariation. As those distributions need not be infinitely divisible, the single period model offers more generality than a continuous-time model.

1.1. Content of the paper, and notation

The paper will give the distributional theory in section 2, and in section 3 establish the CAPM in a fairly stylized way, as the outline should be well known once the building blocks are defined properly. Therefore, the relevant measures of comovement – and thus the *betas* – will be defined and discussed with the distributions in section 2. That section will start with a general discussion, and then spend one subsection for each distribution class in question, where subsection 2.3 will present the novel class. After section 3, section 4 will discuss and conclude.

Let us fix some notation and terminology. We work in $\mathbb{R}^d$ for arbitrary finite $d \geq 2$, and denote by $\mathbb{R}^d_+$ resp. $\mathbb{R}^d_-$ the nonnegative resp. nonpositive closed orthants except allowing for the occational abuse of notation by excluding the origin whenever that point is not interesting. Other sets will be denoted in blackboard bold as well. $\mathbb{S}$ is the unit sphere in $\mathbb{R}^d$. $\mathbb{D}$ is either a pointed cone or $\mathbb{R}^d$. $\mathbb{U}$ is always a subset of the unit simplex, usually intersected with $\mathbb{D}$. Vectors are denoted in boldface, and are columns by default, unless indicated by superscript «$^\top$» (transposition) or given as a gradient. $\mathbf{1}$ is the vector of ones, and $\mathbf{0}$ the null vector. The generic free variable is ξ or (vector) $\boldsymbol{\xi}$. We apply the *signed power* notation $\xi^{<p>} := |\xi|^p \operatorname{sign}(\xi)$ even to vectors, element-wise, e.g $\boldsymbol{\xi}^{<p>} = (\xi_1^{<p>}, \ldots, \xi_d^{<p>})^\top$ (an invertible operation!) Matrices are Greek uppercase slanted bolds (considering the identity $\boldsymbol{I}$ a capital Iota), using non-bold for order 1×1. Random quantities are denoted by upright Latin letters (boldfaced if vector-valued). Minuscles (Greek/Latin, vectors if bold) are either non-random or choice variables. The $\sim$ symbol denotes equal probability law, and Re denotes real part.

2. Distribution classes

Before discussing the distribution classes themselves, let us make a general comment on the key property, namely that the *projections* $\boldsymbol{\xi}^\top \mathbf{X}$ belong to a location–scale class, subject to a suitable constraint on $\boldsymbol{\xi}$. In the history of portfolio theory, this property has been the subject to several mistaken conjectures, most famously the conjecture of Tobin [5] that any two-parameter family would hold. Cass and Stiglitz [18] briefly discussed the property that linear combinations of independent copies of the *multivariate* distribution be in the same class, which is not directly the same as linear combinations of the coordinates – unless the coordinates are independent, and it is perhaps not so well known that Cass and Stiglitz indeed made that reservation to their argument. The pseudo-isotropic class, subsection 2.2, generalizes *symmetric* stability to the property that all projections belong to the same type up to scaling. However, when restricting projections to a cone $\boldsymbol{\xi} \in \mathbb{D}$, the symmetry assumption on the stables can be weakened (subsection 2.1). The class introduced in subsection 2.3 makes the same generalization to this wider class of stables, as pseudo-isotropics make to symmetric stables. In the course, we also generalize certain tools – the covariation measure for co-movement, and the betas for the financial application – from the symmetric stable class.

2.1. Multivariate stable distributions

This subsection largely follows the reference work of Samorodnitsky and Taqqu [19], and several facts will be given without reference.

The stable distributions are precisely the ones attainable through the generalized central limit theorem, and the common defining property of a stable random vector $\mathbf{X}$, is as follows: For two independent copies $\mathbf{X}_1$ and $\mathbf{X}_2$ of $\mathbf{X}$, and any two positive non-random numbers a_1 and a_2, there exist an $a \geq 0$ and a vector $\boldsymbol{d}$, both non-random, such that

$$a_1\mathbf{X}_1 + a_2\mathbf{X}_2 \sim a\mathbf{X} + \boldsymbol{d}.$$

This is sometimes referred to as the *sum-stable* property. As it turns out, there is some parameter $p \in (0, 2]$, called the *index of stability*, for which $a^p = a_1^p + a_2^p$. Commonly, the index is denoted α, but we shall avoid using the first two Greek letters, as they have different meanings in a CAPM context. Also, key to some results are that certain of the properties work out when we replace the index of stability by a different $p \leq 2$, and so we shall use $\wp$ for the index of stability when needed to distinguish – or when two indices are needed.

We shall have more use for the characteristic function; of many representations (see Nolan [20]), we choose the following: There exists a finite *spectral measure* $\varkappa$ on the unit sphere $\mathbb{S} \subset \mathbb{R}^d$ such that

$$\begin{aligned} &\mathsf{E}[\exp(\mathrm{i}\boldsymbol{\theta}^\top \mathbf{X})] = \exp\left(\mathrm{i}\boldsymbol{\theta}^\top \boldsymbol{\mu} - \int_{\mathbb{S}} |\boldsymbol{\theta}^\top \boldsymbol{s}|^p \left[1 + \mathrm{i}\varpi_p(\boldsymbol{\theta}^\top \boldsymbol{s})\right] \varkappa(\mathrm{d}\boldsymbol{s})\right) \\ &\text{where} \quad \varpi_p(\xi) = \tan(-p\pi/2)\,\mathrm{sign}(\xi) \ \text{ for } p \neq 1, \text{ and } \varpi_1(\xi) = \tfrac{2}{\pi}\,\mathrm{sign}(\xi)\ln|\xi|. \end{aligned} \tag{1}$$

$\mathbf{X} - \boldsymbol{\mu}$ is symmetric if $\varkappa$ is symmetric ("iff" for $p < 2$, when also $\varkappa$ is unique). While a multinormal random variable can be written as matrix-transformed iids (plus a constant vector), the non-Gaussian stable laws allow a much richer dependence structures than that; for $p < 2$, only finitely supported $\varkappa$ allow this kind of representation (where the iid case requires the support to be the standard Euclidean unit vectors). For the applications of this paper, one can note that this was not well understood in the early days of modern portfolio theory, where one often made the assumption of matrix-transformed independent coordinates (e.g., the cited works of Fama); on the other hand, there is a rich literature focusing on the *elliptical* special case, where $\varkappa$ is a matrix-transformation of the uniform over $\mathbb{S}$.

The property that makes the portfolio theory work through for a certain subclass of even non-elliptical stable distributions, is the behaviour of linear projections, i.e., linear combinations $\boldsymbol{\xi}^\top \mathbf{X}$. Each such is univariate stable, with distribution fully determined by the following triplet – note the $1_{\{p=1\}}$ in the location – which can be identified by gathering terms in the exponent of (1):

$$\text{scale: } \sigma_{\boldsymbol{\xi}} = \left(\int |\boldsymbol{\xi}^\top \boldsymbol{s}|^p\, \varkappa(\mathrm{d}\boldsymbol{s})\right)^{1/p} \tag{2}$$

$$\text{skewness: } \ = \sigma_{\boldsymbol{\xi}}^{-p} \int |\boldsymbol{\xi}^\top \boldsymbol{s}|^p\,\mathrm{sign}(\boldsymbol{\xi}^\top \boldsymbol{s})\,\varkappa(\mathrm{d}\boldsymbol{s}) \quad \in [-1, 1] \tag{3}$$

$$\text{location: } \mu_{\boldsymbol{\xi}} = \boldsymbol{\xi}^\top \boldsymbol{\mu} - 1_{\{p=1\}} \cdot \tfrac{2}{\pi} \int \boldsymbol{\xi}^\top \boldsymbol{s} \cdot \ln|\boldsymbol{\xi}^\top \boldsymbol{s}|\,\varkappa(\mathrm{d}\boldsymbol{s}). \tag{4}$$

The skewness parameter is often denoted with the letter β; again, this clashes with CAPM notation, the *beta* being one main ingredient in the pricing. Note that this parametrization of scale is a factor of 2 off the standard deviation for the Gaussian – and the parametrizations of skewness have been subject to even more confusion in the literature, dubbed by Hall [21] as a "comedy of errors".

One subclass frequently used in financial modelling is the shifted symmetric class, where $\varkappa$ is antipodally symmetric. This paper shall however have a particular focus on the most asymmetric possible:

Definition 1. *A non-Gaussian multivariate stable random variable is* spectrally negative, *resp.* spectrally positive *if supp* $\varkappa \subseteq \mathbb{R}^d_-$, *resp.* $\subseteq \mathbb{R}^d_+$.

The Gaussians could be included (without mention) whenever suitable. The modelling motivation for focusing on the spectrally negatives, is twofold. First, it is potentially appealing for the modelling of insurance and loans, as the corresponding Lévy process has no positive jumps ("no news is good news", appropriate for a situation where income flows continuously but losses are discrete events). Second, if *no short sale* is allowed, all the positive projections $\{\boldsymbol{\xi}^\top \mathbf{X};\ \boldsymbol{\xi} \in \mathbb{R}^d_+\}$ will have the same skewness, and we are down to a location–scale distribution family. More generally, constant skewness holds on the *polar cone* $\{\boldsymbol{\xi} \in \mathbb{R}^d;\ \boldsymbol{\xi}^\top \boldsymbol{s} \leq 0\ \varkappa\text{-a.s.}\}$ of supp $\varkappa$, and also (with opposite skewness) on the *dual cone* (where $\boldsymbol{\xi}^\top \boldsymbol{s} \geq 0$). We use the letter $\mathbb{D}$ (mnemonic for "dual", in order not to cause confusion with a common notation for probability measure) although the polar cone will be the main focus; for simplicity we state the next lemma for the polar cone only.

Lemma 1. *On the restriction of $\boldsymbol{\xi}$ and $\boldsymbol{\xi}^*$ to the polar cone $\mathbb{D}$ of supp $\varkappa$, the following hold true:*

(a). $\boldsymbol{\xi}^\top \mathbf{X}$ *has skewness equal to* -1.
(b). The upper tail of $\boldsymbol{\xi}^\top \mathbf{X}$ *is light, in that* $\mathsf{E}[\mathrm{e}^{\boldsymbol{\xi}^\top \mathbf{X}}]$ *converges, each* $\boldsymbol{\xi} \in \mathbb{D}$*; if the index of stability is* $\wp < 1$, $\boldsymbol{\xi}^\top \mathbf{X}$ *is supported by the half-line* $(-\infty, \mu_{\boldsymbol{\xi}}]$. *Unless* $\mathbf{X}$ *is Gaussian, the lower tail is heavy, in that* $\max\{0, -\boldsymbol{\xi}^\top \mathbf{X}\}$ *has finite moments only up to (but not including) the index of stability.*
(c). The pair $(\boldsymbol{\xi}^\top \mathbf{X}, \boldsymbol{\xi}^{*\top} \mathbf{X})$ *is bivariate stable and spectrally negative.*
(d). If $\boldsymbol{\xi}^*_{(1)}$ *maximizes location subject to constraining to the polar cone intersected with the σ-unit sphere* $\sigma_{\boldsymbol{\xi}} = 1$*, then we have the equivalence in law*

$$(\sigma_{\boldsymbol{\xi}} \boldsymbol{\xi}^*_{(1)})^\top \mathbf{X} \sim \boldsymbol{\xi}^\top \mathbf{X} + [\mu_{\boldsymbol{\xi}^*_{(1)}} - \mu_{\boldsymbol{\xi}/\sigma_{\boldsymbol{\xi}}}] \sigma_{\boldsymbol{\xi}} \tag{5}$$

so that the random variable $(\sigma_{\boldsymbol{\xi}} \boldsymbol{\xi}^*_{(1)})^\top \mathbf{X}$ first-order stochastically dominates *the random variable* $\boldsymbol{\xi}^\top \mathbf{X}$.

Lemma 1 follows easily from the representation of the chf, see [19], though we give the calculations of part (d) for the 1-stable case:

$$\frac{\mu_{\boldsymbol{\xi}}}{\sigma_{\boldsymbol{\xi}}} = \boldsymbol{\xi}^\top \boldsymbol{\mu} - \frac{2}{\pi} \Big[\int \frac{\boldsymbol{\xi}^\top \boldsymbol{s}}{\sigma_{\boldsymbol{\xi}}} \ln \Big| \frac{\boldsymbol{\xi}^\top \boldsymbol{\sigma}}{\sigma_{\boldsymbol{\xi}}} \Big| \,\mathrm{d}\varkappa + \int \boldsymbol{\xi}^\top \boldsymbol{s} \,\mathrm{d}\varkappa \cdot \frac{\ln \sigma_{\boldsymbol{\xi}}}{\sigma_{\boldsymbol{\xi}}} \Big] = \mu_{\boldsymbol{\xi}/\sigma_{\boldsymbol{\xi}}} + \frac{2}{\pi} \ln \sigma_{\boldsymbol{\xi}} \tag{6}$$

as $\int \boldsymbol{\xi}^\top \boldsymbol{s} \,\mathrm{d}\varkappa / \sigma_{\boldsymbol{\xi}}$ equals the skewness, which is -1. Therefore $\mu_{\sigma_{\boldsymbol{\xi}} \boldsymbol{\xi}^*_{(1)}} - \mu_{\boldsymbol{\xi}} = \sigma_{\boldsymbol{\xi}} \mu_{\boldsymbol{\xi}^*_{(1)}} - \sigma_{\boldsymbol{\xi}} \mu_{\boldsymbol{\xi}/\sigma_{\boldsymbol{\xi}}}$ for the 1-stable case as well. A consequence of part (d) to economic theory, is that there is no agent which maximizes expected nondecreasing utility, who will strictly prefer $\boldsymbol{\xi}^\top \mathbf{X}$ to $(\sigma_{\boldsymbol{\xi}} \boldsymbol{\xi}^*_{(1)})^\top \mathbf{X}$. For the market model in the next section, this property leads to two-fund monetary separation for the market under restriction to $\mathbb{D}$ ([17, Theorem 5.1]) , the essential property for the CAPM market line.

Remark 1 (Sub-stability). *Consider the product* $\mathbf{X}\mathrm{H}$ *where* $\mathbf{X}$ *is p-stable and located at zero, and* H *is an independent positive random variable such that the pth power* H^p *is* $\wp$*-stable with* $\wp < 1$*, located at zero and, necessarily, with skewness of* $+1$*; that is,* $-\ln \mathsf{E}[\mathrm{e}^{\mathrm{i}\vartheta \mathrm{H}^p}] = |\sigma\vartheta|^{\wp}[1 - \mathrm{i} \tan(\wp\pi/2) \operatorname{sign} \vartheta]$. *If* $p\wp \neq 1 \neq p < 2$*, then* $\mathbf{X}\mathrm{H}$ *is stable with index* $p\wp$ *([19, subsection 1.3 and Theorem 2.1.5]). Furthermore,* $\mathbf{X}\mathrm{H}$ *is symmetric iff* $\mathbf{X}$ *is, but spectral negativity is only inherited the same way for* $p < 1$*: If* $\mathbf{X}$ *is spectrally negative,* $p > 1$ *and* $\boldsymbol{\xi} \in \mathbb{R}^d_+$*, then* $\boldsymbol{\xi}^\top \mathbf{X}$ *has skewness of* -1 *and thus a light upper tail, but* $\boldsymbol{\xi}^\top \mathbf{X}\mathrm{H}$ *has both tails equally heavy, as (Hardin [22, p. 4]) its skewness is* $-\frac{\tan([2-p\wp]\cdot\pi/2)}{\tan([2-p]\wp\cdot\pi/2)} \in (-1, 0)$.

Nevertheless the first-order stochastic dominance (5) continues to hold if $\mathbf{X}$ is scaled with some nonnegative H (not necessarily a power of a stable) – with the same $\boldsymbol{\xi}^*_{(1)}$ chosen over the same unit sphere. This motivates that we consider geometric properties that do not directly follow from stability.

2.2. The essentials of pseudo-isotropic random vectors

The pseudo-isotropic distributions (cf. e.g. Jasiulis and Misiewicz [23, Definition 3] or Misiewicz [24] for a reference work) are those multivariate distributions possible in the Eaton [25] problem of finding d-dimensional versions of univariate distributions:

Definition 2. *A symmetric distribution in* $\mathbb{R}^d$ *is called* pseudo-isotropic *if for some order 1 positive-homogeneous* standard $\varsigma : \mathbb{R}^d \to [0, \infty)$ *and some (complex) function g, the characteristic function can be represented by* $q\boldsymbol{\theta} \mapsto \mathsf{E}[\mathrm{e}^{\mathrm{i}q\boldsymbol{\theta}^\top \mathbf{X}}] = \mathrm{e}^{-g(|q|\,\varsigma(\boldsymbol{\theta}))}$. *It is called* pure *if no marginal* X_i *has any point mass at zero.*

Thus, for pure pseudo-isotropic measures, all projections are of the same type: fixing any marginal X_i, then $\boldsymbol{\xi}^\top \mathbf{X} \sim \varsigma(\boldsymbol{\xi}) \mathrm{X}_i$ holds for some order one positive homogeneous $\varsigma \geq 0$, called the *standard* of the distribution. Pseudo-isotropy generalizes ellipticity located at zero (then, ς^2 is a quadratic form) and symmetric stability, and in the same way as the Gaussian admits unique properties among the stables, ellipticity admits properties that are unique among pseudo-isotropics. Pseudo-isotropic distributions can be formed from stable distributions by multiplying by an arbitrary independent univariate variable (without changing ς). This underlies the next definition's focus on the geometry of the ς unit ball rather than, e.g., tail heaviness, cf. also the end of Remark 1. One should note, though, that there are other pseudo-isotropic distributions, see Gneiting [26].

The next definition is a short-form adaptation of a central property studied for pseudo-isotropic distributions, cf. e.g. Koldobsky [27], [28] and coauthors [29] – wherein the reader can also find why the restriction to $p \le 2$ is no loss of generality for $d \ge 3$.

Definition 3. *Let* $\mathbf{X}$ *be pseudo-isotropic. Fix* $p \in [0,2]$*. Suppose that there exists some finite measure* $\varkappa = \varkappa_p$ *supported by the unit sphere, and such that* ς *admits the so-called Blaschke–Lévy representation*

$$\varsigma = \varsigma_p(\boldsymbol{\xi}) = \begin{cases} \left(\int_{\mathbb{S}} |\boldsymbol{\xi}^\top \mathbf{s}|^p \, \varkappa_p(\mathrm{d}\mathbf{s}) \right)^{1/p}, & p \in (0,2] \\ \exp \int_{\mathbb{S}} \ln |A_0 \boldsymbol{\xi}^\top \mathbf{s}| \, \varkappa_0(\mathrm{d}\mathbf{s}), & p = 0, \text{ (some constant } A_0 > 0). \end{cases} \tag{7}$$

If so, we say that ς*, and* $\mathbf{X}$ *and its distribution,* embed in L^p *and that the* p-spectral measure $\varkappa_p$ exists. *Also, we say that the representation* ς_p exists, *hence the subscript. (Other than existence-or-not,* ς *does not depend on* p*.)*

It is a fact that embedding in L^p implies embedding for all indices $\in [0,p)$, and that all pseudo-isotropic random variables embed in L^0 (the Lisitsky [30] conjecture, settled by Koldobsky [28, Corollary 1]) with in the pure case, a uniquely given A_0 computed explicitly in [29, p. 3–4]. The case $p = 2$ – ellipticity – is the only where the dependence structure can be expressed through a matrix transformation standardization (the positive-definite square root of $\int \mathbf{s}\mathbf{s}^\top \, \mathrm{d}\varkappa_2$). The non-elliptical pseudo-isotropic distributions are heavy-tailed: as is well known, non-embedability in L^p ($p \in (0,2]$) implies infinite pth order moment. The interesting case for our purposes is if embedding holds for $p > 1$, in which case ς is strictly quasiconvex and its unit ball a strictly convex set with smooth sphere; by bounded convergence, the gradient $\nabla\varsigma(\boldsymbol{\xi}) = \int (\boldsymbol{\xi}^\top \mathbf{s})^{<p-1>} \mathbf{s}^\top \mathrm{d}\varkappa$ is continuous outside the origin. Portfolio theory for shifted pseudo-isotropic distributions, and consequences of their geometric properties, are given in [14]. The next subsection will generalize to a class unifying (shifted) pseudo-isotropic and spectrally negative random vectors.

2.3. The class $\Psi_{\mathbb{D}}(\varsigma)$ of distributions which satisfy the pseudo-isotropy definition restricted to a cone $\mathbb{D}$

The symmetric stable distributions are purely pseudo-isotropic: the distribution of a projection $\boldsymbol{\xi}^\top \mathbf{X}$ depends only on $\varsigma(\boldsymbol{\xi})$. It is easy to see that on the restriction to the polar cone, or equivalently the restriction to the dual cone, of the support of the spectral measure, the same holds without symmetry, provided that $\boldsymbol{\mu} = \mathbf{0}$; the restriction eliminates the third skeweness parameter. This motivates the unified concept given in Definition 4 below, where the use of dual or polar cone is of course interchangeable.

Definition 4 (The class $\Psi_{\mathbb{D}}(\varsigma)$)**.** *Let* $\mathbb{D}$ *be the dual cone of some subset of* $\mathbb{R}^d$ *and let* ς *be some order 1 homogeneous function* $\varsigma : \mathbb{D} \to [0,\infty)$*. Define* $\Psi_{\mathbb{D}}(\varsigma)$ *to be the class of distributions whose characteristic function* restricted to $\mathbb{D}$ *admits the following representation*

$$\exists g : \mathbb{R} \to \mathbb{C} \quad \text{s.t.} \qquad \mathsf{E}[\mathrm{e}^{\mathrm{i}\boldsymbol{\theta}^\top \mathbf{X}}] = \mathrm{e}^{-g(\varsigma(\boldsymbol{\theta}))} \qquad \text{valid for all } \boldsymbol{\theta} \in \mathbb{D}. \tag{8}$$

The random vector $\mathbf{X}$ *with this distribution is also said to be* $\in \Psi_{\mathbb{D}}(\varsigma)$*. For non-random* $\boldsymbol{\mu}$*, we then call the random vector* $\boldsymbol{\mu} + \mathbf{X}$ shifted $\Psi_{\mathbb{D}}(\varsigma)$*. In either case, we call* ς *its* standard*. We say that* $\mathbf{X}$ *and its distribution are* $\in \Psi_{\mathbb{D}}$ *(resp. shifted* $\Psi_{\mathbb{D}}$*) if there exists some* ς *such that they are* $\in \Psi_{\mathbb{D}}(\varsigma)$ *(resp. shifted* $\Psi_{\mathbb{D}}(\varsigma)$*).*

A random variable which is shifted $\Psi_{\mathbb{D}}(\varsigma)$, and its distribution, and the associated standard ς, are said to embed in L^p *if there exists a finite p*-spectral measure *$\varkappa_p$ supported by the unit sphere, such that the representation* (7) *holds true on $\mathbb{D}$.*

The notation $\Psi_{\mathbb{D}}(\varsigma)$ using the Greek letter Ψ is a mnemonic for ***p**seudo-**i**sotropic on $\mathbb{D}$ with standard* ς, but the author is somewhat reluctant to trying to coin a name for the distribution class. The extension given in Definition 4 appears novel to the author: there is a related concept (called "positive isotropic vectors" by Arias and Koldobsky [31]) where $\mathbb{D} = \mathbb{R}^d_+$ (non-shifted) and imposing the additional condition that $\mathbf{X}$ also a.s. takes values only in $\mathbb{R}^d_+$ – that class includes the spectrally positive stables with index < 1 and location at zero. A negative result is reproduced in part (c) in the next proposition; that negative result is no surprise given the behaviour for the stable case (Remark 1).

Proposition 1. *Assume the random vector $\mathbf{X}$ can* not *be represented as $= \boldsymbol{c}\mathrm{X}$ (where $\boldsymbol{c}$ is a non-random vector and the random variable X is univariate), and that none of its marginals is a Dirac point mass. Then the following hold true.*

(a). If $\mathbf{X}$ is shifted $\Psi_{\mathbb{D}}(\varsigma)$ with ς being strictly quasiconvex, then

$$\nabla\varsigma(\boldsymbol{\xi}) = \int_{\mathbb{S}} \big(\frac{\boldsymbol{\xi}^\top}{\varsigma(\boldsymbol{\xi})}\boldsymbol{s}\big)^{<p-1>}\boldsymbol{s}^\top\, \varkappa_p(\mathrm{d}\boldsymbol{s}) \tag{9}$$

holds in the subgradient sense, and is bounded on the interior of $\mathbb{D}$. If the ς-unit sphere is smooth (which in particular holds under embedding in L^p for some $p > 1$), then $\nabla\varsigma$ is also continuous.

(b). Let $\mathbf{X} \in \Psi_{\mathbb{D}}(\varsigma)$, and H be independent univariate and non-Dirac. Then $\mathbf{X}\mathrm{H} \in \Psi_{\mathbb{D}}(\varsigma)$ as well. In particular, $\mathbf{X}\mathrm{H}$ embeds in L^p if $\mathbf{X}$ does.

(c). A spectrally positive or spectrally negative stable vector, is shifted $\Psi_{\mathbb{R}^n_+}$, and embeds in $\mathsf{L}^\wp$ where $\wp$ is its index of stability. However, if $\wp < 1$, then for every $p > 1$, it does not *embed in L^p, nor does it have finite pth moment.*

(d). Let $\mathbf{X} - \boldsymbol{\mu} \in \Psi_{\mathbb{D}}(\varsigma)$. If $\boldsymbol{\xi}^*_{(1)} \in \operatorname{argmax}_{\boldsymbol{\xi}\in\mathbb{D};\varsigma(\boldsymbol{\xi})=1}\boldsymbol{\xi}^\top\boldsymbol{\mu}$, then we have the equivalence in law

$$(\varsigma(\boldsymbol{\xi})\boldsymbol{\xi}^*_{(1)})^\top\mathbf{X} \sim \boldsymbol{\xi}^\top\mathbf{X} + \varsigma(\boldsymbol{\xi})\big[\boldsymbol{\xi}^*_{(1)} - \frac{\boldsymbol{\xi}}{\varsigma(\boldsymbol{\xi})}\big]^\top\boldsymbol{\mu} \quad \geq \boldsymbol{\xi}^\top\mathbf{X} \tag{10}$$

so that the random variable $(\varsigma(\boldsymbol{\xi})\boldsymbol{\xi}^*_{(1)})^\top\mathbf{X}$ first-order stochastically dominates *the random variable $\boldsymbol{\xi}^\top\mathbf{X}$, this generalizing Lemma 1 part (d).*

Just like in Lemma 1, the last part (d) shall be used to establish two-fund monetary separation under restriction to $\mathbb{D}$ in the next section.

Proof. Part (d) is evident. The nontrivial last statement of part (c) is given in Arias and Koldobsky [31, Corollaries 2.4, 2.5], while the first is clear from Lemma 1. For part (b), write the characteristic function as

$$\mathsf{E}\big[\mathsf{E}[\cos(|\mathrm{H}|\boldsymbol{\theta}^\top\mathbf{X})\big|\mathrm{H}] + \mathrm{i}\,\mathrm{sign}(\mathrm{H})\mathsf{E}[\sin(|\mathrm{H}|\boldsymbol{\theta}^\top\mathbf{X})\big|\mathrm{H}]\big]$$

where both the conditional inner expectations depend $\boldsymbol{\theta}$ only through $\varsigma(|\mathrm{H}|\boldsymbol{\theta}) = |\mathrm{H}|\varsigma(\boldsymbol{\theta})$. Part (a) follows by bounded convergence and homogeneity of degree zero of $\nabla\varsigma$. $\square$

Remark 2. *From part (a), we can extend directional derivatives in directions into $\mathbb{D}$ even on the boundary except at the origin: let $\boldsymbol{\xi}$ and $\boldsymbol{\delta}$ non-null such that both $\boldsymbol{\xi}$ and $\boldsymbol{\xi}+\boldsymbol{\delta}$ both $\in \mathbb{D}$, and define $\nabla\varsigma(\boldsymbol{\xi})\boldsymbol{\delta}$ by taking limits. We shall use the notation $\nabla\varsigma(\boldsymbol{\xi})$ this way to refer to the restriction to $\mathbb{D}\setminus\{0\}$, disregarding any concerns across the boundary and out of $\mathbb{D}$.*

For an example of how the orthant restriction affects the differentiability conditions, let $\hat{\mathrm{X}}_i$ be iid 1-stable with skewness $= -1$, and let $\check{\mathbf{X}}$ be independent 1-stable and elliptic, all located at zero. Consider $\mathbf{X} = \check{\mathbf{X}} +$

$(\hat{X}_1,\ldots,\hat{X}_d)^\top$. Then $\mathbf{X} \in \Psi_{\mathbb{R}^d_+}$, (though not spectrally negative!), ς is strictly convex, and it is C^1 on the orthant (outside the origin). However, the function $\boldsymbol{\xi} \mapsto \sigma_{\boldsymbol{\xi}}$ has nondifferentiabilities where one variable changes sign, so the C^1 property is truly an effect of the orthant restriction.

We can now define a functional which will be denoted $\boldsymbol{\beta}$; it will be precisely the *beta* of the CAPM when we insert the so-called *market portfolio* for the reference vector $\boldsymbol{\xi}^*$. That motivates the restriction to the unit simplex (which will have the interpretation of being the set possible allocation weights in risky portfolios) in the following.

Definition 5. *Let $\mathbf{X} \in \Psi_{\mathbb{D}}(\varsigma)$, with $\mathbb{D}$ such that $\mathbb{U} = \{\boldsymbol{\xi} \in \mathbb{D};\ \mathbf{1}^\top\boldsymbol{\xi} = 1\}$ is nonempty. Suppose that $\varsigma \in \mathsf{C}^1(\mathbb{U})$ and convex on $\mathbb{U}$; in particular this holds under embedding in L^p for some $p > 1$ (for which, in turn, integrable spectrally negative stability is sufficient). Fix some $\boldsymbol{\xi}^*$ interior relative to $\mathbb{U}$. Define $\boldsymbol{\beta} = \frac{1}{\varsigma(\boldsymbol{\xi}^*)}\nabla\varsigma(\boldsymbol{\xi}^*)$ on $\mathbb{U}$, and consider it a linear functional on $\mathbb{U}$: the* beta *of a vector $\boldsymbol{\xi} \in \mathbb{U}$ is defined as*

$$\boldsymbol{\beta}^\top\boldsymbol{\xi} = \frac{1}{\varsigma(\boldsymbol{\xi}^*)}\nabla\varsigma(\boldsymbol{\xi}^*)\boldsymbol{\xi} \qquad \textit{so, in particular,} \qquad \beta_i = \frac{1}{\varsigma(\boldsymbol{\xi}^*)}\cdot\frac{\partial\varsigma}{\partial\xi_i}(\boldsymbol{\xi}^*) \tag{11}$$

β_i is then the beta of investment opportunity number *i. Furthermore, extend through continuity the definition to $\boldsymbol{\xi}^*$ on the boundary under the assumption that $\boldsymbol{\xi}^* + \epsilon\boldsymbol{\xi} \in \mathbb{D}$ for all sufficiently small $\epsilon > 0$.*

If one prefers, one can extend to the entire $\{\mathbb{D};\ \mathbf{1}^\top\boldsymbol{\xi} > 0\}$ by homogeneity of degree zero, equating with $\boldsymbol{\beta}^\top\boldsymbol{\xi}/\mathbf{1}^\top\boldsymbol{\xi}$, as justified above. Note that under embedding in L^p, we can write $\boldsymbol{\beta} = \varsigma(\boldsymbol{\xi}^*)^{-p}\int_{\mathbb{S}} \boldsymbol{s}(\boldsymbol{\xi}^{*\top}\boldsymbol{s})^{<p-1>}\,\varkappa_p(\mathrm{d}\boldsymbol{s})$.

Remark 3. *Remark first that the betas are nonnegative when also $\boldsymbol{\xi} \in \mathbb{D}$; then, all $\boldsymbol{\xi}^\top\boldsymbol{s}$ and $\boldsymbol{\xi}^{*\top}\boldsymbol{s}$ have the same sign. In a pure-jump Lévy process framework (which assumes infinite divisibility, which holds for the stables), the nonnegativity corresponds to the fact that "if all jumps are negative (or all positive), then all simultaneous jumps go in the same direction". However, if $p = 2$ (where the skewness parameter does not enter into the stable case,) this translates all pairs being nonnegatively correlated. When can a coordinate i of beta vanish? Then $\varkappa_p$-a.s. we must have have $0 = (\boldsymbol{e}_i{}^\top\boldsymbol{s})(\boldsymbol{s}^\top\boldsymbol{\xi}^*) \geq |s_i|^2\xi_i^*$, so that $\xi_i^* = 0$ if the ith marginal is non-degenerate. At least in the stable case, it easily follows that even if $\xi_i^* = 0$, the variables $\boldsymbol{\xi}^{*\top}\mathbf{X}$ and X_i must in fact be independent.*

Let us make a few more remarks about the interior vs. boundary of the unit simplex. In the interior, the calculations would make sense under a local C^1 assumption. However, as we are to apply this to a CAPM, it will not make sense unless the Markowitz bullet is a convex set. If for example $\mathbf{X}$ has iid coordinates and an index of stability less than one, we have a "negative diversification effect" and formal calculations may in a certain sense lead to the worst-possible choices with the optimal being non-diversification. In the case of iid 1-stables, we do have convexity of the set, but a piecewise-linear unit sphere, a linear programming problem and then also a corner solution. Embedding in L^p for $p > 1$ is however sufficient to ensure that the problem is well behaved.

2.4. The covariation measure of association

The *beta* is in the elliptical CAPM a covariance-to-variance ratio, and thus defined in terms of a comovement measure. We have seen from the technical side that the *beta*, if required to be unique, is technically tightly connected to continuous differentiability of the ς, for which the existence of moments above order one is but a sufficient condition. This motivates the generalization of a comovement measure commonly applied for symmetric integrable stable distributions only, namely the so-called *covariation.* Samorodnitsky and Taqqu [19, section 2.7] gives two equivalent definitions for this measure of association for bivariate stable vectors, through the two common values

$$\int_{\mathbb{S}} s_1 s_2^{<\wp-1>} \,\mathrm{d}\varkappa(s_1,s_2) = \frac{1}{\wp}\cdot\frac{\mathrm{d}\varsigma^{\wp}(\xi_1^*,\xi_2^*)}{\mathrm{d}\xi_1}\Big|_{\boldsymbol{\xi}^{*\top}=(0,1)} \tag{12}$$

where $\wp$ is the index of stability, commonly (as in [19]) assumed > 1. However, $\wp > 1$ can be replaced by the weaker condition that ς is C^1 outside the origin. In our case, it even suffices that the random variable embeds in L^p for some $p > 1$ in the sense of Definition 4: in that case, $\frac{1}{\wp}\nabla\varsigma^{\wp} = \varsigma^{\wp-1}\nabla\varsigma$, and $\nabla\varsigma$ we have an expression for: Under the assumption of embedding in L^p for $p > 1$, then both sides of (12) are well-defined as long as $\varsigma(0,1) > 0$, and equal

$$\varsigma(0,1)^{\wp-1}\frac{\mathrm{d}\varsigma}{\mathrm{d}\xi_1}(0,1) = \varsigma(0,1)^{\wp-p}\int_{\mathbb{S}} s_1 s_2^{<p-1>}\,\varkappa_p(\mathrm{d}\mathbf{s}). \tag{13}$$

We could take this as an extended definition for the case where a $\wp$-stable embeds for $p > 1$, but it depends directly on the random variable being stable. The following suggested definition does not.

Definition 6 (suggested)**.** *Suppose* $\mathbf{X} \in \Psi_{\mathbb{D}}(\varsigma)$, *with* $\nabla\varsigma$ *bounded and continuous on the unit simplex (which, in particular, holds under embedding in* L^p *for* $p > 1$*). Fix a continuous increasing function* $h : [0,\infty) \mapsto [0,\infty)$ *with* $h(0) = h'(0) = 0$*. Then for* $\boldsymbol{\xi}^* \in \mathbb{D}$ *and* $\boldsymbol{\xi}$ *such that* $\boldsymbol{\xi}^* + \epsilon\boldsymbol{\xi} \in \mathbb{D}$ *for all sufficiently small* $\epsilon > 0$*, and non-random* a *and* a^**, define the* h-covariation of $a + \boldsymbol{\xi}^\top\mathbf{X}$ on $a^* + \boldsymbol{\xi}^{*\top}\mathbf{X}$ *as*

$$[a+\boldsymbol{\xi}^\top\mathbf{X}\,,\; a^*+\boldsymbol{\xi}^{*\top}\mathbf{X}]_h = \frac{h(\varsigma(\boldsymbol{\xi}^*))}{\varsigma(\boldsymbol{\xi}^*)}\nabla\varsigma(\boldsymbol{\xi}^*)\boldsymbol{\xi} \qquad \textit{though} = 0 \textit{ if } \varsigma(\boldsymbol{\xi}^*) = 0. \tag{14}$$

The condition $h(0) = h'(0) = 0$ ensures that the h-covariation tends to zero with $\varsigma(\boldsymbol{\xi}^*)$. With characteristic function $e^{-g(\varsigma(\boldsymbol{\xi}))}$, one can in particular suggest $h = \mathsf{Re}\, g$ or $h = \mathsf{Re}\, g/\mathsf{Re}\, g(1)$ or $h(\sigma) = \mathsf{Re}\,(g(\sigma)/g(1))$, provided $h'(0) = 0$ holds. In that case, it reduces to the covariation as defined in [19] on the common domain:

Lemma 2. *Let* $\mathbf{X}$ *be stable with index* > 1 *and either symmetric (with* $\mathbb{D} = \mathbb{R}^d$*) or spectrally negative (with* $\mathbb{D} = \mathbb{R}^d_+$*). For* $h(\sigma) = \sigma^\alpha$*, then formula* (14) *agrees with* (13)*.*

The lemma does make use of the particular form of the stable characteristic function. However, the h function simply vanishes when we take ratios and calculate betas:

Lemma 3. *We have*

$$\frac{[\boldsymbol{\xi}^\top\mathbf{X}\,,\;\boldsymbol{\xi}^{*\top}\mathbf{X}]}{[\boldsymbol{\xi}^{*\top}\mathbf{X}\,,\;\boldsymbol{\xi}^{*\top}\mathbf{X}]} = \frac{\nabla\varsigma(\boldsymbol{\xi}^*)\boldsymbol{\xi}}{\nabla\varsigma(\boldsymbol{\xi}^*)\boldsymbol{\xi}^*} = \frac{\nabla\varsigma(\boldsymbol{\xi}^*)\boldsymbol{\xi}}{\varsigma(\boldsymbol{\xi}^*)} = \boldsymbol{\beta}^\top\boldsymbol{\xi}. \tag{15}$$

If H *is univariate, independent of* $\mathbf{X}$ *and not a.s. zero, then* $\boldsymbol{\beta}$ *is the same for* $\mathbf{X}\mathrm{H}$ *as for* $\mathbf{X}$*.*
Furthermore, assume $\mathbb{D} \subseteq$ *an orthant and that* $\boldsymbol{\xi} \in \mathbb{D}$*. Then* $\boldsymbol{\beta}^\top\boldsymbol{\xi} \geq 0$*.*

Proof. Positivity is obvious when also $\boldsymbol{\xi} \in \mathbb{D}$. By Proposition 1 part (b), ς does not depend on H. □

The usual caveats concerning covariation do still apply, of course. The covariation of X_1 on X_2 is usually not the same as of X_2 on X_1; sufficient for equality is embedding in L^2 (i.e., ellipticity) or independence (where for stables, we do not have embedding for any p above the index of stability). Furthermore, covariation is additive in the first variable, but commonly not in the second.

There is nothing deep about replacing the commonly taken assumptions of integrability and symmetricity by a mere "as long as well-defined"; the question is rather whether the extension leads to anything useful. Going beyond integrability is nothing novel for elliptical distributions, and embeddability is the sensible geometric sufficient condition that generalizes. Possibly less obvious is how to make sense of the extension to non-symmetric vectors. Here, the betas show that the extension beyond symmetric variables does have an interpretation, as long as one stays within $\mathbb{D}$ (or within $-\mathbb{D}$), where the projections' distributions form a location–scale family.

3. The Capital Asset Pricing Model

An *agent* participating in the market to be given below, will be identified with an initial nonnegative wealth y and a preference ordering consistent with first-order stochastic dominance (that is, we do not assume risk aversion). The preferences themselves will be suppressed in the exposition, as the distributional assumptions will admit two-fund monetary separation, ensuring that all agents choose the same same risky portfolio throughout this section.

The agents face a single period investment decision of allocating wealth y between $\boldsymbol{\xi}$ in $d \in \mathbb{N}$ «risky» investment opportunities, and the remaining $y - \mathbf{1}^\top \boldsymbol{\xi}$ in a numéraire that returns R_f per monetary unit invested, called the *risk-free* return. We split out the numéraire return to write the risky returns vector as $\mathrm{R}_f \mathbf{1} + \boldsymbol{\mu}\mathrm{H}_0 + \mathbf{Z}\mathrm{H}$ for some non-random *location parameter* $\boldsymbol{\mu}$, deliberately using the same notation as the location of a stable vector. Our model is then the *portfolio return*

$$w\mathrm{R}_f + \boldsymbol{\xi}^\top (\boldsymbol{\mu}\mathrm{H}_0 + \mathbf{Z}\mathrm{H}) \tag{16}$$

where we assume *no short sale*, i.e., $\boldsymbol{\xi} \in \mathbb{D} := \mathbb{R}_+^d$, but – as of now – no other restrictions on $\boldsymbol{\xi}$. We could have used a different cone $\mathbb{D}$, but this is the most interesting among the constrained case, and it is general up to a suitable linear transformation. In order to eliminate arbitrage opportunities and degeneracies, assume that

$$\text{some } \mu_i \text{ is} > 0; \qquad \boldsymbol{\xi}^\top (\boldsymbol{\mu}\mathrm{H}_0 + \mathbf{Z}\mathrm{H}) \text{ is not a.s. nonnegative for any } \boldsymbol{\xi} \in \mathbb{D} \setminus \{0\}. \tag{17}$$

although Lemma 4 can do with weaker assumptions.

The presence of R_f, H_0 and H makes room for more general distributions – in particular, H could take both signs – but also helps to illustrate why the index of stability is secondary to the geometric properties of the ς (quasi-) norm:

Lemma 4. *The assumptions that*

$$\mathrm{R}_f \textit{ is independent of everything else, and so is } (\mathrm{H}_0, \mathrm{H}); \qquad \textit{and, } \mathrm{H}_0 \geq 0 \textit{ a.s.} \tag{18}$$

grant the following implication: If $\boldsymbol{\xi}^{*\top}\mathbf{X} \sim \boldsymbol{\xi}^\top\mathbf{X}$ and $(\boldsymbol{\xi}^* - \boldsymbol{\xi})^\top \boldsymbol{\mu} \geq 0$, then the return using $\boldsymbol{\xi}^*$ first-order stochastically dominates the return using $\boldsymbol{\xi}$.

This illustrates why the simplification of taking those three variables to be constant, is as good as without loss of generality; for the more general case, one can easily copy the argument and note what degeneracies one must assume away. We shall therefore make the simplifying assumption that

$$\mathrm{H}_0 = \mathrm{H} = 1, \quad \text{so that the excess returns vector is } \mathbf{X} = \boldsymbol{\mu} + \mathbf{Z}; \quad \text{and, } \mathrm{R}_f = 0. \tag{19}$$

(understanding, if referring to Section 2.1, that we keep the roles from that section; in particular, concerning the role of the shift $\boldsymbol{\mu}$). The key distributional assumptions on the excess returns is then that $\mathbf{Z} \in \Psi_{\mathbb{R}_+^d}$.

Under these assumptions, two-fund monetary separation easily follows from Lemma 4 and the argument of Lemma 1 part (d). Furthermore, strict convexity of the unit ball yields uniqueness:

Lemma 5. *Assume that* (17), (18) *and – for simplicity only –* (19) *hold true. Suppose $\mathbf{Z} \in \Psi_{\mathbb{R}_+^d}(\varsigma)$ with the ς-unit ball being a strictly convex set. Then:*

(a). $\operatorname{argmax}_{\varsigma(\boldsymbol{\xi})=1} \boldsymbol{\xi}^\top \boldsymbol{\mu} =: \boldsymbol{\xi}^*_{(1)}$ *is unique. Then no agent will strictly prefer using* $\boldsymbol{\xi}$ *to using* $\varsigma(\boldsymbol{\xi})\boldsymbol{\xi}^*_{(1)}$*, and unless the two are equal, some agents will strictly prefer to use the latter.*

(b). The Markowitz bullet $\{(\boldsymbol{\xi}^\top \boldsymbol{\mu}, \varsigma(\boldsymbol{\xi}));\ \boldsymbol{\xi} \in \mathbb{E}\}$ *is a strictly convex set.*

The convexity follows by repeating the argument of Lemma 1 to distributions $\in \Psi_{\mathbb{D}}$; the convexity of the Markowitz bullet follows by noting that the problem of minimizing scale subject to excess return (and to $\mathbf{1}^\top \boldsymbol{\xi} = 1$), is a convex problem with convex value function.

The pricing in the Capital Asset Pricing Model can be derived under the *assumption* of trade-off between excess return ($\boldsymbol{\xi}^\top \boldsymbol{\mu}$, desired) against some dispersion functional $\varsigma = \varsigma(\boldsymbol{\xi})$ (which will be a scalar multiple of scale in the setting of Lemma 1, when the trade-off is in fact justified). From this on, the arguments can be copied from a text-book with minor modifications; we mention that the common approach assumes risk aversion, while this exposition sticks to merely a preference of more to less (first-order stochastic dominance). Still, we merely sketch the derivation: Every agent holds the same (up to scaling) location–dispersion-efficient risky portfolio, which has to be the market portfolio. Let us disregard those investment opportunities which are not undertaken; they will be in zero demand because the price is higher than implied by the pricing formula. Starting with $\boldsymbol{\xi}^*$, some non-null scaling of the market portfolio, an agent can consider to buy in addition a (sufficiently small) vector $\boldsymbol{\varepsilon}$, and scale up or down the exposure in $\boldsymbol{\xi}^*$ as to maintain total level of dispersion. Define thus the function b implicitly by $\varsigma(\boldsymbol{\varepsilon} + (1 - b(\boldsymbol{\varepsilon})/\varsigma(\boldsymbol{\xi}^*))\boldsymbol{\xi}^*) = \varsigma(\boldsymbol{\xi}^*)$. Implicitly differenting wrt. $\boldsymbol{\varepsilon}$ and inserting for $\boldsymbol{\varepsilon} = \mathbf{0}$ yields $\nabla b(\mathbf{0}) = \nabla\varsigma(\boldsymbol{\xi}^*)$. By the assumed efficiency, $\boldsymbol{\varepsilon} = \mathbf{0}$ must maximize location given dispersion, yielding the formal first-order condition $\boldsymbol{\mu}^\top = (\boldsymbol{\mu}^\top \boldsymbol{\xi}^*/\varsigma(\boldsymbol{\xi}^*))\nabla b(\mathbf{0}) = \frac{\boldsymbol{\mu}^\top \boldsymbol{\xi}^*}{\varsigma(\boldsymbol{\xi}^*)}\nabla\varsigma(\boldsymbol{\xi}^*)$. This determines the excess returns ($\boldsymbol{\mu}$) as a vector $\boldsymbol{\beta}$ scaled by the excess return on the market portfolio: the individual investment opportunities are priced according to their marginal *relative* contribution $\boldsymbol{\beta}^\top = \frac{1}{\varsigma(\boldsymbol{\xi}^*)}\nabla\varsigma(\boldsymbol{\xi}^*)$ – to the risk of the market portfolio. The market portfolio is commonly expressed in *weights*, accommodated by scaling (necessitating that it is not a free good, which is uncontroversial for the application).

We summarize, noting that the embedding condition is sufficient to justify the use of a first-order condition:

Proposition 2 (CAPM for spectrally negative opportunities)**.** *Assume that* (17), (18) *and – for simplicity only –* (19) *hold true. Suppose* $\mathbf{Z} \in \Psi_{\mathbb{R}^d_+}(\varsigma)$ *with* ς *being convex and* C^1 *with bounded derivative on the unit simplex (this is stronger than all parts of Lemma 5, but in particular, it holds under embedding in* L^p *for some* $p > 1$*).*

Then all investment opportunities in strictly positive demand are priced so that their excess returns vector $\boldsymbol{\mu}$ *satisfies* $\boldsymbol{\mu}^\top = \frac{\boldsymbol{\mu}^\top \boldsymbol{\xi}^*}{\varsigma(\boldsymbol{\xi}^*)}\nabla\varsigma(\boldsymbol{\xi}^*)$*, for the unique-up-to-scaling efficient portfolio* $\boldsymbol{\xi}^*$*. This formula also upper bounds the return (lower bounds the price) for investment opportunities in zero demand.*

Assuming furthermore that $\boldsymbol{\xi}^*$ *is not a free good* ($\mathbf{1}^\top \boldsymbol{\xi}^* \neq 0$)*, we can without loss scale it down to the market portfolio of weights of risky opportunities (so that* $\mathbf{1}^\top \boldsymbol{\xi}^* = 1$*); the* betas *of the investment opportunities in positive demand, are then given by formula* (11) *of Definition 5 and are all strictly positive if* $\mathbb{D} = \mathbb{R}^d_+$*.*

The latter strict positivity holds by positive demand, i.e., $\xi^*_i > 0$, and arguing as Lemma 3. Once $\boldsymbol{\xi}^*$ is scaled to unity price, $\boldsymbol{\mu}^\top \boldsymbol{\xi}^*$ translates into $\mathrm{R}_m + \mathrm{R}_f$ in CAPM lingo. The result also justifies our extensions of covariation, as β_i of an opportunity i in positive demand, is its covariation on the market, divided by the market's covariation on itself, like in the known integrable (shifted) symmetric-stable framework.

Among further properties known from elliptial CAPM, is the notion that in a well-diversified market, one will only get paid to take on systematic risk. (Of course, the term "risk" here means dispersion, having translated zero location, so it is not to say that one does not get paid for expected losses.) Suppose opportunity number i contributes to the spectral measure by a point mass at $\boldsymbol{e}_i$, but apart from this point mass, the $s_i \neq 0$ subset has zero measure – both for X_i as well as any other marginal. As long as opportunity number i is of negligible weight in the market portfolio, the point mass at $\boldsymbol{e}_i$ makes no contribution to the beta. This reservation is of course substantial even for the elliptical CAPM, as a non-infinitesimal increment changes the market portfolio; the apparent difference for $p < 2$ is that the infinitesimality reservation enters already when the beta is *defined*, rather than when it is applied and interpreted.

Not all properties carry over nicely though. In elliptical portfolio theory, two-fund separation carries over even to the case without a risk-free opportunity being accessible, leading to Black's so-called zero-beta CAPM. There is no such two-fund separation result in the non-elliptical symmetric case with unconstrained portfolios ([14, Theorem 11]), not even on the positive orthant, and thus none in the case of this paper. For the same reason, we can not hope for a partially hedgeable non-market income to be hedged by a single fund.

4. A brief discussion

The foundational portfolio theory is probabilistic in nature, and does largely ignore objectionable properties like violation of limited liability for the Gaussian – and for the stable distributions as well, except the positive ones of infinite positive mean. On the other hand, there is an extensive literature on the empirical fit (or empirical testability!) of the CAPM. We shall not enter the latter discussion. Rather, we shall discuss some stylized properties and point out plausibly desirable or questionable model features. Symmetric-stable distributions have been a part of portfolio theory for over fifty years, so what does the focus on spectrally negative stables, and the generalizations, bring to the table?

As already mentioned, the positive projections of a spectrally negative stable vector will have only the lower tail heavy; the upper tail will be light, and the sample paths of a corresponding Lévy process will have no upwards jumps. Those asymmetry properties can be seen as appealing from an insurance and loans point of view[1]; even though there might be a bound on single-exposure losses, then portfolio losses could still be heavy tailed to beyond the insolvency of the insurer/lender. That is not to say that the sample paths are linearly upper bounded; under integrability, or assuming embedding for $p > 1$, the continuous part is of infinite variation, a consequence of the martingale property of the zero-location part. The further generalization to $\Psi_{\mathbb{D}}$, gives a class of distributions which need not share the properties of the spectrally negative stables; One obvious feature is that the class includes distributions with both tails heavy (e.g., $\mathbf{X}$H where $\mathbf{X}$ is spectrally negative stable and H takes values ± 1 with probabilities $\pi_{\pm} > 0$ and 0 with probability $1 - \pi_{+} - \pi_{-}$, stable only if $\pi_{+} + \pi_{-} < 1$). Furthermore, the question of infinite divisibility is out in the unknown, and with it, the question of whether we can build a continuous-time model with independent increments.

In addition to the obvious tail heaviness issues, the model of course has other questionable features. The absence of "sudden good news" for the spectrally negative model is disputable for a general market model (although that has not precluded the use of Gaussian continuous-time models where there is no discontinuous arrival of information at all). Why worry – from a technical point of view – about the general market and not focus on a market segment where the assumptions are more fitting? Because the model would then have to make either an assumption of stochastic independence between the segments, or the assumption that the agents cannot invest across segments. Concerning the latter, suppose for example that the markets are jointly stable, but the other segment has symmetric returns; then the agents will have to determine their skewness as well, and the optimum is not trivial except the elliptical case. Finally, let us discuss the assumption of no short sale: one can arguably defend non-shorting of, e.g., retail insurance or loan exposures. On the other hand, one can argue that insurance – with any sort of deductible – means that some agents are taking positions

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[1] For a flavour of both types of loss events, one can see Moscadelli [32], who studies *operational losses* (due to failed internal processes, fraud, infrastructure failures etc.) in the *banking sector.* The estimated tail index is significantly heavier than two for all eight business lines considered, heavier than one in three of them, and statistically tied to one (inside a 95 percent confidence interval) in the other five.

opposite of others; even still, the betas would characterize the demand of those agents who cannot take short positions.

In any case, we have given a proper generalization of elliptical and symmetric-stable CAPM, which can thus be seen as yet another robustification of the well-established model.

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