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ALGORITHM FOR ZEROS OF MONOTONE MAPS IN BANACH SPACES

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Abstract: Let E be a p-uniformly convex real Banach space with uniformly G$\hat{a}$teaux differentiable norm such that $\frac{1}{p}+\frac{1}{q}=1, p \geq 2$ and E^* its dual space. Let $A : E \to E^*$ be a bounded and t-strongly monotone mapping such that $A^{-1}0 \neq \emptyset$. We introduce an explicit iterative algorithm that converges strongly to the unique point $x^* \in A^{-1}0$ in arbitrary real Banach spaces.

Keywords: G$\hat{a}$teaux differentiable norm; Strongly monotone; Uniformly convex.

MSC: 47H06, 47H09, 47J05, 47J25.

1. Introduction

Let E be a real Banach space and let E^* be the dual space of E. We study the method of approximating the zeros of a nonlinear equation of the form

$$0 \in Au, \tag{1}$$

where $u \in E$ and $A : E \to 2^{E^*}$ is a monotone operator. This is a general form for problems of minimization of a function, variational inequalities and so on. Let E be a real normed space of dimension ≥ 2 and let $S := \{x \in E : \|x\| = 1\}$. E is said to have a G$\hat{a}$teaux differentiable norm (or E is called smooth) if the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each $x, y \in S$; E is said to have a uniformly G$\hat{a}$teaux differentiable norm if for each $y \in S$ the limit is attained uniformly for $x \in S$. Further, E is said to be uniformly smooth if the limit exists uniformly for $(x, y) \in S \times S$.
The modulus of convexity of E, $\delta_E : (0, 2] \to [0, 1]$ is defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| = \|y\| = 1, \|x-y\| > \epsilon\right\}.$$

E is uniformly convex if and only if $\delta_E(\epsilon) > 0$ for every $\epsilon \in (0, 2]$. Let $p > 1$, then E is said to be p-uniformly convex if there exists a constant $c > 0$ such that $\delta_E(\epsilon) \geq c\epsilon^p$ for all $\epsilon \in (0, 2]$. Observe that every p-uniformly convex space is uniformly convex. A normed linear space E is said to be strictly convex if

$$\|x\| = \|y\| = 1, x \neq y \Rightarrow \frac{\|x+y\|}{2} < 1.$$

Every uniformly convex space is strictly convex.

Lemma 1. *(See e.g., Chidume [16], p. 135): Let $E = \ell_p$. Then for $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and for each pair $x, y \in E$, the following inequalities hold:*

$$\|\frac{1}{2}(x+y)\|^q + \|\frac{1}{2}(x-y)\|^q \leq \left[2^{-1}\left(\|x\|^p + \|y\|^p\right)\right]^{q-1}, \textit{ for } 1 < p \leq 2, \tag{2}$$

and

$$\|x+y\|^q + \|x-y\|^q \leq 2^{-1}\left(\|x\|^p + \|y\|^p\right), \textit{ for } 2 \leq p < \infty. \tag{3}$$

We use Lemma 1 to give example of uniformly convex spaces.

Example 1. ℓ_p $(1 < p < \infty)$ *spaces are uniformly convex.*

Proof. Given $\epsilon \in (0, 2]$, let $x, y \in \ell_p$ be such that $\|x\| \leq 1, \|y\| \leq 1$ and $\|x - y\| \geq \epsilon$. Two cases arise.
Case 1: $1 < p \leq 2$. In this case, inequality (2) yields:

$$\begin{aligned} \|\frac{1}{2}(x+y)\|^q + \|\frac{1}{2}(x-y)\|^q &\leq \left[2^{-1}\left(\|x\|^p + \|y\|^p\right)\right]^{q-1} \\ &\leq 2^{-(q-1)}2^{q-1} = 1. \end{aligned}$$

Thus, $\|\frac{1}{2}(x+y)\|^q \leq 1 - \|\frac{1}{2}(x-y)\|^q \leq 1 - \left(\frac{\epsilon}{2}\right)^q$ such that $\|\frac{x+y}{2}\| \leq \left[1 - \left(\frac{\epsilon}{2}\right)^q\right]^{1/q} < 1$.
Therefore, by choosing $\delta = 1 - \left[1 - \left(\frac{\epsilon}{2}\right)^q\right]^{1/q} > 0$, we obtain $\|\frac{x+y}{2}\| \leq 1 - \delta$. Thus

$$\delta_E(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2}\right\} = \delta > 0,$$

which shows that ℓ_p $(1 < p \leq 2)$ are uniformly convex.
Case 2: $2 < p < \infty$. The result follows as in case 1 by using inequality (3). Indeed,

$$\|x+y\|^q + \|x-y\|^q \leq 2^{-1}(2) = 1,$$

$$\|\frac{x+y}{2}\|^q \leq \frac{1}{2^q}\left(1 - \epsilon^q\right) = \frac{1}{2^q} - \left(\frac{\epsilon}{2}\right)^q,$$

$$\|\frac{x+y}{2}\| \leq \left[\frac{1}{2^q} - \left(\frac{\epsilon}{2}\right)^q\right]^{1/q} < 1.$$

Therefore, by choosing $\delta = 1 - \left[\frac{1}{2^q} - \left(\frac{\epsilon}{2}\right)^q\right]^{1/q} > 0$, we obtain $\|\frac{x+y}{2}\| \leq 1 - \delta$. Thus

$$\delta_E(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2}\right\} = \delta > 0,$$

which shows that ℓ_p $(2 < p < \infty)$ are uniformly convex. $\square$

Let E be a real normed space and let $J_p, (p > 1)$ denote the generalized duality mapping from E into 2^{E^*} given by

$$J_p(x) = \left\{f \in E^* : \langle x, f\rangle = \|x\|^p, \|f\| = \|x\|^{p-1}\right\},$$

where E^* is the dual space of E and $\langle .,. \rangle$ denotes the generalized duality pairing. It is well known (see, for example, Xu [31]) that $J_p(x) = \|x\|^{p-2} J_2(x)$ if $x \neq 0$. For $p = 2$, the mapping J_2 from E to 2^{E^*} is called normalized duality mapping.

Remark 1. *Let E be a uniformly convex Banach space and E^* its dual space. The following properties of the normalized duality map have been established (see e.g [20], [30], [31], [32]):*

(i) if E is smooth, then J_p is single-valued;
(ii) if E is reflexive, then J_p is onto;
(iii) if E is smooth, strictly convex, and reflexive, then $J_p^* : E^* \to 2^E$ is the generalized duality map from E^* to E;
(iv) if E has uniform Gâteaux differentiable norm, then J_p is norm-to-weak* uniformly continuous on bounded sets.

Let E be a p-uniformly convex real Banach space and $A : E \to E^*$ be a single-valued map. The map A is said to be:

(i) monotone if for each $x, y \in E$, we have

$$\langle x - y, Ax - Ay \rangle \geq 0;$$

(ii) $t-$strongly monotone if there exist a constant $t > 0$ such that for each $x, y \in E$, we have

$$\langle x - y, Ax - Ay \rangle \geq t\|x - y\|^p.$$

Monotone operators have turned out to be ubiquitous in modern optimization and analysis (see, e.g., [4], [6], [26], [27]). Interest in monotone operators stems mainly from their usefulness in numerous applications. Consider, for example (see e.g Chidume et al. [8]), the following: Let $f : E \to \mathbb{R}$ be a proper and convex function. The subdifferential of f at $x \in E$ is defined by

$$\partial f(x) = \{x^* \in E^* : f(y) - f(x) \geq \langle y - x, x^* \rangle \; \forall y \in E\}.$$

Monotonicity of $\partial f : E \to 2^{E^*}$ on E can be easily verified, and that $0 \in \partial f(x)$ if and only if x is a minimizer of f . Setting $\partial f = A$, it follows that solving the inclusion $0 \in Au$ in this case, is the same as solving for a minimizer of f. Several existence theorems have been established for the equation $Au = 0$ when A is of the monotone-type (see e.g., Deimling [21]; Pascali and Sburlan [24]).
A single-valued map $A : E \to E$ is called accretive if for each $x, y \in E$, there exists $j_2(x - y) \in J_2(x - y)$ such that

$$\langle j_2(x - y), Ax - Ay \rangle \geq 0.$$

In a Hilbert space, the normalized duality map is the identity map. Hence, in Hilbert spaces, monotonicity and accretivity coincide.

There have been extensive research efforts on inequalities in Banach spaces and their applications to iterative methods for solutions of nonlinear equations of the form $Au = 0$. Assuming existence, for approximating a solution of $Au = 0$, where A is of accretive-type, Browder [5] defined an operator $T : E \to E$ by $T := I - A$, where I is the identity map on E. He called such an operator pseudo-contractive. It is trivial to observe that zeros of A correspond to fixed points of T. For Lipschitz strongly pseudo-contractive maps, Chidume [18] proved the following theorem.

Theorem 1. *(Chidume [18]:) Let $E = L_p, 2 \leq p < \infty$, and $K \subset E$ be nonempty closed convex and bounded. Let $T : K \to K$ be a strongly pseudocontractive and Lipschitz map. For arbitrary $x_1 \in K$, let a sequence $\{x_n\}$ be defined iteratively by*

$$x_{n+1} = (1 - \lambda_n)x_n + \lambda_n T x_n, n \in \mathbb{N},$$

where $\lambda_n \in (0,1)$ satisfies the following conditions:

(i) $\sum_{n=1}^{\infty} \lambda_n = \infty$,

(ii) $\sum_{n=1}^{\infty} \lambda_n^2 < \infty$.

Then, $\{x_n\}$ converges strongly to the unique fixed point of T.

The above theorem has been generalized and extended in various directions, leading to flourishing areas of research, for the past thirty years or so, for numerous authors (see e.g., Censor and Reich [7]; Chidume [9], [18], [10]; Chidume and Bashir [12]; Chidume and Chidume [13], [14]; Chidume and Osilike [15] and a host of other authors).

Recent monographs emanating from these researches include those by Berinde [3], Chidume [11], Goebel and Reich [17], and William and Shahzad [29].

However, it occurs that most of the existing results on the approximation of solutions of monotone-type maps have been proved in Hilbert spaces. Unfortunately, as has been rightly observed, many and probably most mathematical objects and models do not naturally live in Hilbert spaces. The remarkable success in approximating the zeros of accretive-type mappings is yet to be carried over to equations involving nonlinear monotone mappings in general Banach spaces. Perhaps, part of the difficulty in extending the existing results on the approximation of solutions of accretive-type mappings to general Banach spaces is that, since the operator A maps E to E^*, the recursion formulas used for accretive-type mappings may no longer make sense under these settings. Take for instance, if x_n is in E, Ax_n is in E^* and any convex combination of x_n and Ax_n may not make sense. Moreover, most of the inequalities used in proving convergence theorems when the operators are of accretive-type involve the normalized duality mappings which also appear in the definition of accretive operators.

Recently, Diop et al [22] introduced an iterative scheme and proved the following strong convergence theorem for approximation of the solution of equation $Au = 0$ in a 2-uniformly convex real Banach space. In particular, they proved the following theorem.

Theorem 2. *Diop et al [22]: Let E be a 2-uniformly convex real Banach space with uniformly Gâteaux differentiable norm and E^* its dual space. Let $A : E \to E^*$ be a bounded and k-strongly monotone mapping such that $A^{-1}0 \neq \emptyset$. For arbitrary $x_1 \in E$, let $\{x_n\}$ be the sequence defined iteratively by:*

$$x_{n+1} = J_2^{-1}(J_2 x_n - \alpha_n A x_n), n \in \mathbb{N}, \tag{4}$$

where J_2 is the normalized duality mapping from E into E^* and $\{a_n\} \subset (0,1)$ is a real sequence satisfying the following conditions:

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$.

Then, there exists $\gamma_0 > 0$ such that if $\alpha_n < \gamma_0$, the sequence $\{x_n\}$ converges strongly to the unique solution of the equation $Ax = 0$.

It is our purpose in this paper to introduce an explicit iterative algorithm that converges strongly to the solution of equation (1) in real Banach spaces. Furthermore, we obtained as corollary, the theorems of Diop et al. [22] for $p = 2$ and Chidume et al. [8] for $E := L_p, 1 < p < \infty$ and $\lambda_n = \lambda\ \forall\ n \in \mathbb{N}$, $\lambda \in (0,1)$.

2. Preliminaries

In the sequel, we shall need the following definitions and results.

Definition 1. *Let E be a smooth real Banach space with the dual E^*.*

(i) The function $\phi_1 : E \times E \to \mathbb{R}$ is defined by

$$\phi_1(x,y) = \|x\|^2 - 2\langle x, J_2 y\rangle + \|y\|^2, \text{ for all } x, y \in E, \tag{5}$$

where J_2 is the normalized duality map from E to E^* (see e.g, Alber [1]).

(ii) The function $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(x,y) = p\left(q^{-1}\|x\|^q - \langle x, J_p y\rangle + p^{-1}\|y\|^p\right), \text{ for all } x, y \in E, \tag{6}$$

where J_p is the generalized duality map from E to E^* and such that $\frac{1}{p} + \frac{1}{q} = 1$, $q \geq p \geq 2$.

(iii) The map $V : E \times E^* \to \mathbb{R}$ is defined by

$$V(x,x^*) = p\left(q^{-1}\|x\|^q - \langle x, x^*\rangle + p^{-1}\|x^*\|^p\right) \ \forall \ x \in E, x^* \in E^*. \tag{7}$$

Remark 2. *These remarks follow from Definition 1:*

(i) If $E = H$, a real Hilbert space, then equation (5) reduces to $\phi_1(x,y) = \|x-y\|^2$ for $x, y \in H$. It is obvious from the definition of the function ϕ_1 that

$$(\|x\| - \|y\|)^2 \leq \phi_1(x,y) \leq (\|x\| + \|y\|)^2 \text{ for all } x, y \in E. \tag{8}$$

(ii) For $p = 2$, $\phi(x,y) = \phi_1(x,y)$. Also, it is obvious from the definition of the function ϕ that

$$(\|x\| - \|y\|)^p \leq \phi(x,y) \leq (\|x\| + \|y\|)^p \text{ for all } x, y \in E. \tag{9}$$

(iii) It is obvious that

$$V(x,x^*) = \phi(x, J_p^{-1}x^*) \ \forall \ x \in E, x^* \in E^*. \tag{10}$$

Theorem 3. *Xu [31]: Let $p > 1$ be a fixed real number and E be a real Banach space. The following are equivalent:*

(i) E is p-uniformly convex;

(ii) there is a constant $c_1 > 0$ such that for all $x, y \in E$ and $j_p(x) \in J_p(x)$,

$$\|x+y\|^p \geq \|x\|^p + p\langle y, J_p(x)\rangle + c_1\|y\|^p;$$

(iii) there is a constant $c_2 > 0$ such that

$$\langle x - y, j_p(x) - j_p(y)\rangle \geq c_2\|x-y\|^p, \forall \ x, y \in X \text{ and } j_p(x) \in J_p(x), j_p(y) \in J_p(y).$$

Lemma 2. *Kamimura and Takahashi [23]: Let E be a smooth uniformly convex real Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences from E. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\phi(x_n, y_n) \to 0$ as $n \to \infty$, then $\|x_n - y_n\| \to 0$ as $n \to \infty$.*

Lemma 3. *Tan and Xu [28]: Let $\{\alpha_n\}$ be a sequence of non-negative real numbers satisfying the following relation:*

$$\alpha_{n+1} \leq \alpha_n + \sigma_n, n \geq 0$$

such that $\sum_{n=0}^{\infty} \sigma_n < \infty$. *Then* $\lim_{n\to\infty} \alpha_n$ *exists. If in addition, the sequence* $\{\alpha_n\}$ *has a subsequence that converges to* 0. *Then* $\{\alpha_n\}$ *converges to* 0.

Lemma 4. *Zălinescu [32]. Let* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *be increasing with* $\lim_{t\to\infty} \psi(t) = \infty$. *Then* J_ψ^{-1} *is single-valued and uniformly continuous on bounded sets if and only if E is a uniformly convex Banach space.*

3. Main Result

We first give and prove the following lammas which are useful in establishing our main results.

Lemma 5. *Let E be a reflexive strictly convex and smooth real Banach space with* E^* *as its dual. Then,*

$$V(x, x^*) + p\left\langle J_p^{-1}x^* - x, y^*\right\rangle \le V(x, x^* + y^*) \tag{11}$$

for all $x \in E$ *and* $x^*, y^* \in E^*$.

Proof.

$$\frac{1}{p}V(x, x^* + y^*) = \left(q^{-1}\|x\|^q - \langle x, x^* + y^*\rangle + p^{-1}\|x^* + y^*\|^p\right) \tag{12}$$

$$\frac{1}{p}V(x, x^*) = \left(q^{-1}\|x\|^q - \langle x, x^*\rangle + p^{-1}\|x^*\|^p\right) \tag{13}$$

$$\begin{aligned}
\frac{1}{p}V(x, x^* + y^*) - \frac{1}{p}V(x, x^*) &= -\langle x, y^*\rangle + p^{-1}\|x^* + y^*\|^p - p^{-1}\|x^*\|^p \\
&= \langle -x, y^*\rangle + p^{-1}\|x^* + y^*\|^p - p^{-1}\|x^*\|^p \\
&= \left\langle J_p^{-1}x^* - x, y^*\right\rangle - \left\langle J_p^{-1}x^*, y^*\right\rangle \\
&\quad + p^{-1}\|x^* + y^*\|^p - p^{-1}\|x^*\|^p \\
&\ge \left\langle J_p^{-1}x^* - x, y^*\right\rangle - \left\langle J_p^{-1}x^*, y^*\right\rangle \\
&\quad + p^{-1}\left(\|x^*\|^p + p\left\langle y^*, J_p^{-1}x^*\right\rangle + d_p\|y^*\|^q\right) - p^{-1}\|x^*\|^p \\
&= \left\langle J_p^{-1}x^* - x, y^*\right\rangle + p^{-1}d_p\|x^*\|^p \\
&\ge \left\langle J_p^{-1}x^* - x, y^*\right\rangle
\end{aligned}$$

Thus, $V(x, x^*) + p\left\langle J_p^{-1}x^* - x, y^*\right\rangle \le V(x, x^* + y^*)$. □

Lemma 6. *Let E be a uniformly convex Banach space. The duality mapping* $J_p^{-1} : E^* \to E$ *is Lipschitz on every bounded set in* E^**; that is, for all* $R > 0$, *there exists a positive constant L such that*

$$\|J_p^{-1}(x^*) - J_p^{-1}(y^*)\| \le L\|x^* - y^*\|,$$

for all $x^*, y^* \in E^*$ *with* $\|x^*\| \le R$ *and* $\|y^*\| \le R$.

Proof. From Lemma 4, J_p^{-1} is uniformly continuous on bounded subsets of E^* implies that for all $R > 0$, there exists a nondecreasing function ψ_0 with $\psi_{0^+}(0) = 0$ such that

$$\|J_p^{-1}(x^*) - J_p^{-1}(y^*)\| \le \psi_0(\|x^* - y^*\|),$$

for all $x^*, y^* \in E^*$ with $\|x^*\| \leq R$ and $\|y^*\| \leq R$. By taking $\psi_0(\|x^* - y^*\|) := L\|x^* - y^*\|$, the result follows. $\square$

Lemma 7. *For $p > 1$, let E be a p-uniformly convex real Banach space. For $d > 0$, let $B_d(0) := \{x \in E : \|x\| \leq d\}$. Then for arbitrary $x, y \in B_d(0)$,*

$$\|x - y\|^p \geq \phi(x, y) + d\|x\|^p - \frac{p}{q}\|x\|^q, \;\; q \geq p \geq 2. \tag{14}$$

Proof. Since E is a p-uniformly convex space, then by condition (ii) of Theorem 3, for any $x, y \in B_d(0)$, we have that

$$\|x + y\|^p \geq \|x\|^p + p\left\langle y, J_p(x)\right\rangle + d\|y\|^p.$$

Replacing y by $-y$ gives

$$\|x - y\|^p \geq \|x\|^p - p\left\langle y, J_p(x)\right\rangle + d\|y\|^p.$$

Interchanging x and y and simplifying by p, we get

$$\begin{aligned} p^{-1}\|x - y\|^p &\geq p^{-1}\|y\|^p - \left\langle x, J_p(y)\right\rangle + dp^{-1}\|x\|^p \\ &= q^{-1}\|x\|^q - \left\langle x, J_p(y)\right\rangle + p^{-1}\|y\|^p - q^{-1}\|x\|^q + dp^{-1}\|x\|^p \\ &= p^{-1}\phi(x, y) - q^{-1}\|x\|^q + dp^{-1}\|x\|^p, \end{aligned}$$

so that

$$\phi(x, y) \;\leq\; p\left(p^{-1}\|x - y\|^p \;+\; q^{-1}\|x\|^q \;-\; dp^{-1}\|x\|^p\right),$$

which is equivalent to

$$\|x - y\|^p \geq \phi(x, y) + d\|x\|^p - \frac{p}{q}\|x\|^q,$$

establishing the lemma. $\square$

Theorem 4. *Let E be a p-uniformly convex real Banach space with uniformly Gâteaux differentiable norm such that $\frac{1}{p} + \frac{1}{q} = 1, q \geq p \geq 2$ and E^* its dual space. Let $A : E \to E^*$ be a bounded and t-strongly monotone mapping such that $A^{-1}0 \neq \emptyset$. For arbitrary $x_1 \in E$, let $\{x_n\}$ be the sequence defined iteratively by:*

$$x_{n+1} = J_p^{-1}(J_p x_n - \lambda_n A x_n), n \in \mathbb{N}, \tag{15}$$

where J is the generalized duality mapping from E into E^* and $\{\lambda_n\} \subset (0, \gamma_0), \gamma_0 < 1$ is a real sequence satisfying the following conditions:

(i) $\sum_{n=1}^{\infty} \lambda_n = \infty;$

(ii) $\sum_{n=1}^{\infty} \lambda_n^2 < \infty.$

Then, the sequence $\{x_n\}$ converges strongly to the unique point $x^* \in A^{-1}0$.

Proof. Let $x^* \in E$ be a solution of the equation $Ax = 0$. There exists $r > 0$ sufficiently large such that:

$$r \geq max\left\{4\left(\frac{p}{q}\|x^*\|^q - d\|x^*\|^p\right), \phi(x_1, x^*)\right\} \text{ and } \gamma_0 := \min\left\{1, \frac{ptr}{4M_0}\right\}. \tag{16}$$

We divide the proof into two steps.
Step 1: We prove that $\{x_n\}$ is bounded. It suffices to show that $\phi(x^*, x_n) \leq r, \forall \;\; n \in \mathbb{N}$. The proof is by induction. By construction, $\phi(x^*, x_1) \leq r$. Assume that $\phi(x^*, x_n) \leq r$ for some $n \in \mathbb{N}$. We show

that $\phi(x^*, x_{n+1}) \leq r, \forall\ \ n \in \mathbb{N}$.
From inequality(9), we have $\|x_n\| \leq r^{\frac{1}{p}} + \|x^*\|$. Since A is bounded, define

$$M_0 := pL \sup \left\{ \|Ax_n\|^2 : \|x_n\| \leq r^{\frac{1}{p}} + \|x^*\| \right\}, \tag{17}$$

where L is a Lipschitz constant of J_p^{-1} and $p \geq 2$. We compute as follow by using the definition of x_{n+1}:

$$\begin{aligned}
\phi(x^*, x_{n+1}) &= \phi\left(x^*, J_p^{-1}(J_p x_n - \lambda_n A x_n)\right) \\
&= V\left(x^*, J_p x_n - \lambda_n A x_n\right) \\
&\leq V(x^*, J_p x_n) - p\lambda_n \left\langle J_p^{-1}(J_p x_n - \lambda_n A x_n) - x^*, Ax_n - Ax^* \right\rangle \\
&\quad \text{(by Lemma (5) with } y^* {=} \lambda_n A x_n) \\
&= \phi(x^*, x_n) - p\lambda_n \langle x_n - x^*, Ax_n - Ax^* \rangle \\
&\quad - p\lambda_n \left\langle J_p^{-1}(J_p x_n - \lambda_n A x_n) - J_p^{-1}(J_p x_n), Ax_n \right\rangle.
\end{aligned}$$

Using the strong monotonicity of A, Schwartz inequality and Lipschitz property of J_p^{-1}, we obtain

$$\begin{aligned}
\phi(x^*, x_{n+1}) &\leq \phi(x^*, x_n) - pt\lambda_n \|x_n - x^*\|^p \\
&\quad + q\lambda_n \|J_p^{-1}(J_p x_n - \lambda_n A x_n) - J_p^{-1}(J_p x_n)\| \|Ax_n\| \\
&\leq \phi(x^*, x_n) - pt\lambda_n \|x_n - x^*\|^p + p\lambda_n^2 L \|\|Ax_n\|^2 \\
&\leq \phi(x^*, x_n) - pt\lambda_n \left(\phi(x^*, x_n) + d\|x^*\|^p - \frac{p}{q}\|x^*\|^q \right) + \lambda_n^2 M_0 \\
&\quad \text{(using \ Lemma 7)} \\
&\leq \phi(x^*, x_n) - pt\lambda_n \phi(x^*, x_n) + pt\lambda_n \left(\frac{p}{q}\|x^*\|^q - d\|x^*\|^p \right) + \lambda_n \gamma_0 M_0 \\
&\leq r - pt\lambda_n r + pt\lambda_n \frac{r}{4} + pt\lambda_n \frac{r}{4} \\
&= \left(1 - \frac{pt\lambda_n}{2}\right) r \\
&< r.
\end{aligned}$$

Hence, $\phi(x^*, x_{n+1}) \leq r$. By induction, $\phi(x_*, x_n) \leq r \ \ \forall \ \ n \in \mathbb{N}$. Thus, from inequality (9), $\{x_n\}$ is bounded.
Step 2: We now prove that $\{x_n\}$ converges strongly to the unique point $x^* \in A^{-1}0$. Following the same arguments as in step 1, the boundedness of $\{x_n\}$ and that of A, there exists a positive constant M_0 such that

$$\phi(x^*, x_{n+1}) \leq \phi(x^*, x_n) - pt\lambda_n \|x_n - x^*\|^p + \lambda_n^2 M_0. \tag{18}$$

Consequently, $\phi(x^*, x_{n+1}) \leq \phi(x^*, x_n) + \lambda_n^2 M_0$.
By the hypothesis that $\sum_{n=0}^{\infty} \lambda_n^2 < \infty$ and Lemma 3, we have that $\lim_{n\to\infty} \phi(x^*, x_n)$ exists. From inequality (18), we have $\sum_{n=0}^{\infty} \lambda_n \|x_n - x^*\| < \infty$. Using the fact $\sum_{n=0}^{\infty} \lambda_n = \infty$, it follows that $\liminf \|x_n - x^*\|^p = 0$. Consequently, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x^*$ as $k \to \infty$. Since $\{x_n\}$ is bounded and J_p is norm-to-weak* uniformly continuous on bounded subset of E, it follows that $\{\phi(x^*, x_n)\}$ has a subsequence that converges to 0. Therefore, by Lemma 2, $\{\phi(x^*, x_n)\}$ converges strongly to 0. Also, by Lemma 2, $\|x_n - x^*\| \to 0$ as $n \to \infty$. $\square$

Corollary 1. *Diop et al. [22]: Let E be a 2-uniformly convex real Banach space with uniformly Gâteaux differentiable norm and E^* its dual space. Let $A : E \to E^*$ be a bounded and k-strongly monotone mapping such that $A^{-1}0 \neq \emptyset$. For arbitrary $x_1 \in E$, let $\{x_n\}$ be the sequence defined iteratively by:*

$$x_{n+1} = J_2^{-1}(J_2 x_n - \alpha_n A x_n), n \in \mathbb{N}, \tag{19}$$

where J_2 is the normalized duality mapping from E into E^* and $\{a_n\} \subset (0,1)$ is a real sequence satisfying the following conditions:

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$.

Then, there exists $\gamma_0 > 0$ such that if $\alpha_n < \gamma_0$, the sequence $\{x_n\}$ converges strongly to the unique solution of the equation $Ax = 0$.

Proof. By taking $p = 2$, the proof follows from Theorem 4. $\square$

Corollary 2. *Chidume et al [8]: Let $E = L_p, 1 < p < \infty$. Let $A : E \to E^*$ be a strongly monotone and Lipschitz map. For $x_1 \in E$ arbitrary, let the sequence $\{x_n\}$ be defined by:*

$$x_{n+1} = J_2^{-1}(J_2 x_n - \lambda A x_n), n \in \mathbb{N}, \tag{20}$$

where $\lambda \in (0, \delta)$. Then, the sequence $\{x_n\}$ converges strongly to $x^* \in A^{-1}(0)$ and x^* is unique.

Proof. Take $E = L_p, 1 < p < \infty$ and $\lambda_n = \lambda$, the proof follows from Theorem 4. $\square$

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