Amenable Banach Algebras Of p-Compact Operators

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Abstract

Let X be a Banach space. A Banach operator algebra $\mathcal{U}(X)$ is said to be amenable if every continuous derivation from $\mathcal{U}(X)$ into its dual Banach bimodules is inner. We study this notion, via a newly defined symmetric approximation property, in the Banach operator ideal of p-compact operators modelled on specific Banach spaces.

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AMENABLE BANACH ALGEBRAS of p-COMPACT OPERATORS

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Abstract: Let X be a Banach space. A Banach operator algebra $\mathcal{U}(X)$ is said to be amenable if every continuous derivation from $\mathcal{U}(X)$ into its dual Banach bimodules is inner. We study this notion, via a newly defined symmetric approximation property, in the Banach operator ideal of p-compact operators modelled on specific Banach spaces.

Keywords: p-compact ideal; dual p-compact ideal; minimum p-summing ideal.

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1. Introduction

The amenability question for algebras $\mathcal{K}(X)$ (of compact operators on a given Banach space X) was first considered in 1972 by Johnson [4]. Since then, many researchers have studied this notion in the closed subalgebras of $\mathcal{B}(X)$, the algebra of bounded linear operators on X. However, little is known about the amenability of general Banach operator ideals in relation to the geometric properties of X.

It is important that such investigations are made. For instance, amenability is characterized in C^*-algebras as nuclearity [8, Theorem 5.6.73]. Do we have such parallels in general Banach operator ideals? Solutions to similar questions may serve as transport theorems between the geometric theory of a given Banach space X and the Banach algebraic theory of $\mathcal{U}(X)$ for some operator ideal $\mathcal{U}$. This is one of the motivations for this article.

This paper is organized as follows: In section 1 we give a short motivation, and recall basic definitions from the theory of tensor products of Banach spaces, operator ideals, operator theory, and linear functional analysis. We shall use these definitions throughout this article. In section 2 we recollect few preliminary results from literature and fix some notations. We define the property $(\mathbb{A}_p)$ in Section 3 and present the main results of the paper. We conclude with an open question in Section 4. The solution to this question serves as a concrete proof of the nonequivalence of $(\mathbb{A}_p)$ and $(\mathbb{A})$.

Let $\mathcal{U}$ be a Banach algebra, and let X be a Banach $\mathcal{U}$-bimodule. A bounded linear map $D : \mathcal{U} \longrightarrow X$ which satisfies the equality:

$$D(ab) = D(a) \cdot b + a \cdot D(b) \qquad (a, b \in \mathcal{U}),$$

is called a bounded derivation. All maps of the form $a \mapsto a \cdot x - x \cdot a$ $(a \in \mathcal{U})$, for a fixed $x \in X$, are bounded derivations. Such derivations are called inner. $\mathcal{U}$ is said to be amenable if every derivation from $\mathcal{U}$ into its dual Banach bimodules is an inner derivation. Equivalent characterizations of amenability are given in Theorem 2 below.

Let X be a banach space, and identify finite rank operators $\mathcal{F}(X)$ and $X' \otimes X$ in the usual way, where we associate to each element $v = \sum_i \lambda_i \otimes x_i \in X' \otimes X$ the operator $\bar{v} \in \mathcal{F}(X)$ defined by $\bar{v}(x) = \sum_i \lambda_i(x) x_i$ $(x \in X)$. We shall particularly talk about the κ_p^d-norm of an element of $X' \otimes X$, and the projective norm of an operator in $\mathcal{F}(X)$, denoted by $\|\cdot\|_\wedge$. The completion of $X' \otimes X = \mathcal{F}(X)$ in the projective norm is the *tensor algebra of* X and it is denoted by $X' \hat{\otimes} X$. Moreover, the tensor algebra is a Banach algebra ([3, II.2.19]).

An *operator algebra* (*operator ideal*) on X is a subalgebra (an ideal) of $\mathcal{B}(X)$ containing $\mathcal{F}(X)$. An operator algebra (operator ideal) $\mathcal{U}$ on X is a *Banach operator algebra* (*Banach operator ideal*) if it is a Banach algebra with respect to some norm. A *closed operator algebra* (*closed operator ideal*) is an operator algebra (operator ideal) which is closed in $(\mathcal{B}(X), \|\cdot\|)$.

A Banach space X has the *approximation property* (AP) [*compact approximation property* (CAP)] if, for each compact set $K \subset X$ and each $\epsilon > 0$, there exists $T \in \mathcal{F}(X)$ $[T \in \mathcal{K}(X)]$ with $\|Tx - x\| < \epsilon$ $(x \in K)$. Suppose further that there is a constant $m > 0$ (independent of K and ϵ) such that T can be chosen with $\|T\| \leq m$. Then X has the *bounded approximation property* (BAP) [*bounded compact approximation property* $(BCAP)$].

Let $p \geq 1$, and $K \subset X$ be a subset of a Banach space X. K is said to be relatively p-compact if there is a strongly p-summable sequence (x_n) in X such that for every $k \in K$ there exists $(\lambda_n) \in \ell_{p'}$ such that $k = \sum_{n=1}^\infty \lambda_n x_n$. We say that a linear operator $T : X \longrightarrow Y$ is p-compact if T maps the closed unit ball of X into a relatively p-compact subset in Y. Let $\mathcal{K}_p(X, Y)$ be the set of all p-compact operators from X to Y. Then $\mathcal{K}_p(X, Y)$ is a Banach space with a suitable factorization norm κ_p and $(\mathcal{K}_p, \kappa_p)$ is a Banach operator ideal [6, Theorem 4.2]. It follows from [9, Chapter 9] that the dual $(\mathcal{K}_p^d, \kappa_p^d)$ of $(\mathcal{K}_p, \kappa_p)$ is also a Banach operator ideal.

Denote by tr_X the usual trace on $X' \otimes X$, namely, the unique bounded linear functional on $X' \otimes X$ which is defined on elementary tensors by

$$tr_X(\lambda \otimes x) = \lambda(x), (\lambda \in X', x \in X)$$

(also see [2, §2.5]).

Let X and Y be normed spaces, and let $T \in \mathcal{B}(X, Y)$. Then there exists a unique $T' \in \mathcal{B}(Y', X')$ such that

$$\langle Tx, \lambda \rangle = \langle x, T'\lambda \rangle \quad (x \in X, \lambda \in Y').$$

We call T' the *adjoint* of T. Furthermore, $\|T'\| = \|T\|$, and the map

$$T' : (Y', \sigma(Y', Y)) \longrightarrow (X', \sigma(X', X))$$

is continuous.

Given a closed subspace $\mathcal{U}$ of $\mathcal{B}(X, Y)$, define

$$\mathcal{U}^a = \{T' \in \mathcal{B}(Y', X') : T \in \mathcal{U}\}. \tag{1.1}$$

Then the map $T \longrightarrow T'$, $\mathcal{U} \longrightarrow \mathcal{U}^a$ is an isometric linear bijection. Now let $\mathcal{U}$ be a closed operator algebra in $\mathcal{B}(X)$. Then $\mathcal{U}^a$ is a closed operator algebra in $\mathcal{B}(X')$. It is clear that $\mathcal{U}$ has a bounded right approximate identity if and only if $\mathcal{U}^a$ has a bounded left approximate identity. We would like to remark that if X has a basis and (P_n) is a sequence of natural projections, so that $(P_n) \subset \mathcal{F}(X)$, the (P_n) is a sequential bounded left approximate identity in $\mathcal{K}(X)$ with the bound of the approximate identity being the basis constant.

2. Preliminary results

We put on record the following theorems and propositions whose proofs can be consulted in [8] and [11].

Proposition 1. [11, Proposition 3.2] and [8, Proposition 1.9.20]. *Let G be any irreducible $n \times n$ matrix group, and put*

$$d = \frac{1}{|G|} \sum_{x \in G} x \otimes x^{-1}.$$

Then d is the unique element of $\mathbb{M}_n \otimes \mathbb{M}_n$ which is a diagonal for both $\mathbb{M}_n$ and $\mathbb{M}_n^{op}$, and

$$d = \frac{1}{n} \sum_{i,j=1}^{n} E_{ij} \otimes E_{ji}.$$

Theorem 1. [8, Theorem 2.9.37]. *Let X be a non-zero Banach space.*

1. *The algebra $\mathcal{A}(X)$ $[\mathcal{K}(X)]$ has a bounded left approximate identity if and only if X has BAP [BCAP].*
2. *Suppose that X has BAP [BCAP]. The null sequences in $\mathcal{A}(X)$ $[\mathcal{K}(X)]$ factor.*
3. *The algebra $\mathcal{A}(X)$ has a bounded right approximate identity if and only if X' has BAP.*
4. *Suppose that $\mathcal{K}(X)$ has a bounded right approximate identity. Then X' has BCAP.*
5. *For each closed operator ideal $\mathcal{U}$ in $\mathcal{B}$ with $\mathcal{U} \subset \mathcal{K}(X)$, $\mathcal{U}$ has a bounded approximate identity if and only if $\mathcal{U}$ has a bounded right approximate identity.*

Theorem 2. (Helemskii, Johnson; [8, Theorem 2.9.65]). *Let A be a Banach algebra. Then the following conditions on A are equivalent:*

1. *A is amenable;*
2. *A has an approximate diagonal;*
3. *A has a virtual diagonal;*
4. *A has a bounded approximate identity, and $\mathcal{H}^1(A, I_\pi'') = \{0\}$;*
5. *A has a bounded approximate identity and A is biflat;*
6. *A has a bounded approximate identity and I_π'' has a bounded right approximate identity.*

Write $X \simeq Y$ (respectively, $X \cong Y$), if X and Y are isomorphic (respectively, isometric) normed spaces. Let $d(X, Y)$ be the Banach-mazur distance between X and Y, that is, the infimum of the set $\{\|T\|\|T^{-1}\|\}$, where T is an isomorphism between X and Y. We will denote by $\overline{X}$ the completion of a normed space X. Given the vectors $x_1, x_2, ..., x_r$ of some linear space X, we will denote by $\mathrm{sp}\{x_1, x_2, ..., x_r\}$ their linear span. If $\mathcal{U}$ is a normed linear algebra and X is a $\mathcal{U}$-bimodule, we will denote by $\mathcal{Z}(X)$ the center of X.

Let $\mathcal{U}$ be a dense subalgebra of $(X' \otimes X, \|\cdot\|_\wedge)$(hence of $X' \hat{\otimes} X$). Define $\mathcal{U}_1 := (\mathcal{U}, \|\cdot\|_\wedge)$ and $\mathcal{U}_2 := (\mathcal{U}, \kappa_p^d)$, where $1 \leq p \leq \infty$. Then $\hat{\mathcal{U}_1} \cong X' \hat{\otimes} X$ and $\hat{\mathcal{U}_2} \cong \Pi_p^{min}(X)$ by [7, Remark 2.7]. Since the finite rank operators are (absolutely) p-summing, it follows from [7, Theorem 2.6] that for every finite rank operator $S = \sum_{i \leq n} x_i^* \otimes y_i$, we have

$$\begin{aligned} \kappa_p^d(S) \leq \sum_{i \leq n} \kappa_p^d(x_i^* \otimes y_i) &= \sum_{i \leq n} \pi_p(x_i^* \otimes y_i) \\ &\leq \sum_{i \leq n} \|x_i^*\|\|y_i\|, \end{aligned}$$

and on taking the infimum on both sides we obtain

$$\kappa_p^d(S) \leq \|S\|_\wedge. \tag{2.1}$$

On the other hand, $\|T\| \leq \kappa_p(T)$ for every p-compact operator T by [6, p. 22]. Since S' is also a finite rank operator, it is p-compact. Hence

$$\begin{aligned} \|S\| &= \|S'\| \\ &\leq \kappa_p(S') \\ &= \kappa_p^d(S) \end{aligned} \tag{2.2}$$

by the definition of a dual ideal. In fact, Eqn(1.3) is true for any dual p-compact operator T as well; that is,

$$\|T\| \leq \kappa_p^d(T) \tag{2.3}$$

for any dual p-compact operator T. Together, (2.1) and (2.2) yield

$$\|S\| \leq \kappa_p^d(S) \leq \|S\|_\wedge. \tag{2.4}$$

Lemma 1. ([10, Lemma 2.4]). *Let Y and Z be Banach spaces. If $T : Y \longrightarrow Z^*$ is a weakly compact operator and $R := T^*_{|Z}$, then $R^{**} = T^*$.*

Theorem 3. ([7, Theorem 4.5]). *Let $1 \leq p \leq \infty$. A Banach space X has the approximation property of type p (AP_p, for short) if and only if*

$$\overline{\mathcal{F}(Y,X)}^{\kappa_p^d} = \mathcal{K}_p^d(Y,X),$$

for every Banach space Y.

3. Algebra of p-compact operators and amenability

In this section, we present some results for Banach spaces with property $(\mathbb{A}_p)$. We begin by recalling important definitions from which our definition of the aforesaid property emerges.

Let X be a linear space. A *biorthogonal system* of size n for X is a pair $((x_1, \dots, x_n), (\lambda_1, \dots, \lambda_n))$ in $X^{(n)} \times X'^{(n)}$ such that $\langle x_i, \lambda_j \rangle = \delta_{i,j}$ $(i, j \in \mathbb{N}_n)$. Each system defines a homomorphism

$$\Phi : \mathbb{M}_n \longrightarrow \mathcal{F}(X) : (\alpha_{i,j})_{i,j} \mapsto \sum_{i,j=1}^{n} \alpha_{i,j}\lambda_j \otimes x_i.$$

In fact, $\forall a = (a_{ij}), b = (b_{ij}) \in \mathbb{M}_n$ and $x \in X$, it holds that $ab = (\sum_k a_{ik}b_{kj})$ and

$$\begin{aligned} [\Phi(a) \circ \Phi(b)](x) &= [\sum_{i,k} a_{ik}\lambda_k \otimes x_i] \circ [\sum_{l,j} b_{lj}\lambda_j \otimes x_l](x) \\ &= \sum_{i,k} a_{ik}\lambda_k(\sum_{l,j} b_{lj}\lambda_j(x)x_l)x_i \\ &= \sum_{i,k}(\sum_{j} a_{ik}b_{kj}\lambda_j(x))x_i \\ &= [\sum_{i,j}(\sum_{k} a_{ik}b_{kj})\lambda_j \otimes x_i](x) \\ &= \Phi(\sum_{k} a_{ik}b_{kj})(x). \end{aligned}$$

Therefore, $\Phi(ab) = \Phi(a)\Phi(b)$.

The identity in $\mathbb{M}_n$ is denoted by E_n. Clearly $\Phi(E_n)$ is an idempotent in $\mathcal{F}(X)$ and a projection onto a subspace of X.

A *symmetric approximation property* for X, which implies that $\mathcal{K}(X)$ is amenable, was defined by Grønbæk, Johnson and Willis in [11, Definition 4.1] and the following variant is given in [8, Definition 5.6.62]:

Let X be a Banach space. Then X is said to have property $(\mathbb{A})$ if there is a directed set A such that, for each $\alpha \in A$, there exists $n_\alpha \in \mathbb{N}$, a biorthogonal system of size n_α with corresponding homomorphism Φ_α, and an irreducible $n_\alpha \times n_\alpha$ matrix group G_α and such that:

$(\mathbb{A})$(i) $(\Phi_\alpha(E_{n_\alpha}) : \alpha \in A)$ is a bounded approximate identity for $\mathcal{A}(X)$,
$(\mathbb{A})$(ii) $\sup\{\|\Phi_\alpha(x)\| : x \in G_\alpha, \alpha \in A\} < \infty$.

Set $P_\alpha = \Phi_\alpha(E_{n_\alpha})$ $(\alpha \in A)$. It is proved in [8, Theorem 5.6.63], a theorem which is attributed to N. Grønbæk, B.E. Johnson and G.A. Willis ([11, Theorem 4.2]), that the foregoing property implies that the Banach operator algebra $\mathcal{K}(X)$ is amenable.

Remark 1. *Condition $(\mathbb{A})(i)$, above, is equivalent to the following conditions*
(iii) $P_\alpha(x) \to x$ $(x \in X)$, and
(iv) $P'_\alpha(\lambda) \to \lambda$ $(\lambda \in X')$.

Proposition 2. *If condition $(\mathbb{A})(i)$ above holds, then X has the bounded approximation property.*

Proof. This follows from the first assertion of Theorem 1. □

We define here the intended *symmetric approximation property of type p*, $1 \le p \le \infty$, as follows:

Definition 1. *Let X be a Banach space. Then X is said to have property $(\mathbb{A}_p)$ if there is a directed set A such that, for each $\alpha \in A$, there exists $n_\alpha \in \mathbb{N}$, a biorthogonal system of size n_α, $((x_i)_{i\le n_\alpha}, (\lambda_j)_{j\le n_\alpha}) \in \ell^s_p(X) \times \ell^w_{p'}(X')$, with corresponding homomorphism Φ_α, and an irreducible $n_\alpha \times n_\alpha$ matrix group G_α and such that:*
$(\mathbb{A}_p)$(i) $(\Phi_\alpha(E_{n_\alpha}) : \alpha \in A)$ is a bounded approximate identity for $\Pi^{min}_p(X)$,
$(\mathbb{A}_p)$(ii) $\sup\{\kappa^d_p(\Phi_\alpha(x)) : x \in G_\alpha, \alpha \in A\} < \infty$.
For $p = \infty$, the above definition is replaced by that of Grønbæk, Johnson and Willis in [11, Definition 4.1].

The next arguments arise immediately from the above definition.

Since $P_\alpha = \sum_{i\le n_\alpha} x^*_i \otimes x_i$ is a finite rank operator on X, its dual has the form $P'_\alpha = \sum_{i\le n_\alpha} \hat{x}_i \otimes \lambda_i$, and so it is p-compact as the following computation shows: for each α, the triangle inequality and the fact that $\kappa_p(\cdot)$ is an ideal norm give

$$\begin{aligned}
\kappa_p(P'_\alpha) &\le \sum_{i\le n_\alpha} \|\hat{x}_i\|\|\lambda_i\| \\
&= \sum_{i\le n_\alpha} \|x_i\|\|\lambda_i\| \\
&= \sup_{\|x\|\le 1} \sum_{i\le n_\alpha} \|x_i\||\langle x, \lambda_i\rangle| \qquad (3.1)\\
&\le \sup_{\|x\|\le 1} (\sum_{i\le n_\alpha} \|x_i\|^p)^{1/p} (\sum_{i\le n_\alpha} |\langle x, \lambda_i\rangle|^{p'})^{1/p'} \qquad \text{(Hölder's inequality)}\\
&= \|(\lambda_i)\|^w_{p'} \|(x_i)\|^s_p.
\end{aligned}$$

Hence, for each α, it holds that

$$\kappa^d_p(P_\alpha) \le \|(\lambda_i)\|^w_{p'} \|(x_i)\|^s_p, \tag{3.2}$$

and so each P_α has a finite $\kappa^d_p(\cdot)$-norm. That is, $P_\alpha \in \Pi^{min}_p(X)$.

Our assumption about the membership of (x_i) and (λ_i) in the above definition as well as the finiteness of the κ_p^d-norm on P_α should not come as a surprise. It has been proved in [5, Theorem 3.3] that the operator ideal $(\mathcal{K}_p, \kappa_p)$ is associated to the tensor norm $/d_p$, for $1 \leq p < \infty$, where d_p is the Chevet-Saphar norm defined by

$$d_p(u) = \inf\{\|(x_n)_n\|_{p'}^w \|(y_n)_n\|_p^s\},$$

where the infimum is taken over all the possible representations of $u = \sum_{j=1}^n x_j \otimes y_j$. We denote by $/d_p$ the left injective tensor norm associated to d_p. We have that $/d_p = g'_{p'}$ (see for instance [12, Theorem 7.20]), and hence $/d_p = (g^*_{p'})^t$, where $\alpha^* = (\alpha')^t$ for a tensor norm α, and α^* is called the *contragradient* (or *adjoint*) of a tensor norm α.

Moreover, it holds that for any Banach spaces X and Y, $\mathcal{K}_p^{min}(X, Y) \overset{1}{=} X' \overset{/d_p}{\otimes} Y$ ([5, Remark 3.8]). Since Y has the κ_p-approximation property if and only if $\mathcal{K}_p^{min}(X, Y) \overset{1}{=} \mathcal{K}_p(X, Y)$ for every Banach space X (cf. [5, p. 11]), we have that for each α, $\kappa_p(P'_\alpha) \leq \sum_{i \leq n_\alpha} \|x_i\|\|\lambda_i\| < \infty$ and in our case, whenever $Y = X$ has the κ_p-approximation property, then on identifying a tensor product with the operator it represents,

$$\kappa_p(P'_\alpha) = /d_p(P'_\alpha) \leq d_p(P'_\alpha) \leq \|(\lambda_i)\|_{p'}^w \|(x_i)\|_p^s, \tag{3.3}$$

which recovers Eqn(3.2) above.

The question of uniform bound of net (P_α) in α is of utmost importance and is the next one to be addressed.

Since $(x_i)_i \in \ell_p^s(X)$ in the definition of the property $\mathbb{A}_p$, it is clear that for each $x^* \in X'$, $\sum_i |\langle x^*, x_i\rangle|^p < \infty$ and that

$$\sup_{x^* \in B_{X'}} (\sum_i |\langle x^*, x_i\rangle|^p)^{1/p} < \infty,$$

too. In fact, we have that for any $x^* \in X'$

$$(\sum_i |\langle x^*, x_i\rangle|^p)^{1/p} \leq (\sum_i \|x_i\|^p)^{1/p} \|x^*\|.$$

This says that $\ell_p^s(X)$ is a linear subspace of $\ell_p^w(X)$. Since the standard unit vector basis in $\ell_{p'}$ (resp. c_0 if $p = 1$) is always a weak ℓ_p sequence in $\ell_{p'}$ (resp. c_0), the inclusion of $\ell_p^s(X)$ in $\ell_p^w(X)$ is strict, unless X is finite dimensional.

Proceeding from Eqn(3.1) we have

$$\begin{aligned}
\kappa_p^d(P_\alpha) &\leq \sup_{\|x\|\leq 1} (\sum_{i \leq n_\alpha} \|x_i\| |\langle x, \lambda_i\rangle|) \\
&= \sup_{\|x^*\|\leq 1, \|x\|\leq 1} (\sum_{i \leq n_\alpha} |\langle x^*, x_i\rangle| |\langle x, \lambda_i\rangle|) \\
&\leq \sup_{\|x^*\|\leq 1, \|x\|\leq 1} (\sum_{i \leq n_\alpha} |\langle x^*, x_i\rangle|^p)^{1/p} (\sum_{i \leq n_\alpha} |\langle x, \lambda_i\rangle|^{p'})^{1/p'} \\
&= \sup_{\|x\|\leq 1} (\sum_{i \leq n_\alpha} |\langle x, \lambda_i\rangle|^{p'})^{1/p'} \sup_{\|x^*\|\leq 1} (\sum_{i \leq n_\alpha} |\langle x^*, x_i\rangle|^p)^{1/p} \\
&= \|(\lambda_i)_i\|_{p'}^w \|(x_i)_i\|_p^w.
\end{aligned}$$

It follows that

$$\kappa_p^d(P_\alpha) \leq \|(\lambda_i)_i\|_{p'}^w \|(x_i)_i\|_p^w. \tag{3.4}$$

Clearly, Eqn(3.4) offers a better estimate of the $\kappa_p^d(\cdot)$-norm of P_α than Eqn(3.2).

We will prove that our definition of the property $(\mathbb{A}_p)$ implies that the Banach algebra $\mathcal{K}_p^d(X)$ of dual p-compact bounded linear operators on X is amenable. Although we will follow the method of [8, Theorem 5.6.63] (also see [11]), it must be pointed out that our result, namely Theorem 4 below, is a *'p-level'* version of [8, Theorem 5.6.63] and this is, consequently, a special case when $p = \infty$ since $\mathcal{K}_\infty^d(X) = \mathcal{K}_\infty = \mathcal{K}(X)$.

We observe that if condition $(\mathbb{A}_p)$(í) holds, then for any $T \in \Pi_p^{min}(X)$ it holds that
(v) strong-$\lim_\alpha P_\alpha T = T =$ strong-$\lim_\alpha TP_\alpha$.
Let $T \in \Pi_p^{min}(X)$. Then there exists a sequence (T_n) in $\mathcal{F}(X)$ such that $T = \kappa_p^d - \lim_n T_n$. It follows from Eqn(1.4) that

$$\|T - T_n\| \leq \kappa_p^d(T - T_n),$$

where $\|\cdot\| = \|\cdot\|_{op}$, since $T - T_n \in \mathcal{B}(X)$. Hence $T = \|\cdot\| - \lim_n T_n$, and so $T \in \mathcal{A}(X)$. Therefore $\Pi_p^{min}(X) \subseteq \mathcal{A}(X)$ boundedly. Hence condition (v) implies that condition (iii) and (iv) are satisfied, whence condition $(\mathbb{A})$(i) follows. Therefore condition $(\mathbb{A}_p)$(í) implies condition $(\mathbb{A})$(i) as might be expected from the fact that $\Pi_p^{min}(X) \subseteq \mathcal{A}(X)$ boundedly as normed spaces. Thus condition $(\mathbb{A}_p)$(í) implies, via Proposition 3.6, that X has bounded approximation property. Of course, $(\mathbb{A}_p)$(íí) implies $(\mathbb{A})$(ii) by Eqn(1.4).

Alternatively, it follows from Eqn(2.3) and the fact that $\Pi_p^{min}(X) \subseteq \mathcal{A}(X)$ that $(\mathbb{A}_p)$(í) and $(\mathbb{A}_p)$(íí) imply $(\mathbb{A})$(i) and $(\mathbb{A})$(ii), respectively, so that the symmetric approximation property of type p (namely $\mathbb{A}_p$) implies the (classical) symmetric approximation property (namely $\mathbb{A}$). Whence, by Proposition 2 again, X has bounded approximation property.

Remark 2. *Since $\|\cdot\|$ and $\kappa_p^d(\cdot)$ are not equivalent, $(\mathbb{A})$ does not imply $(\mathbb{A}_p)$, and this very fact makes $(\mathbb{A}_p)$ worthy of study in relation to amenability at the p-level.*

Proposition 3. *If condition $(\mathbb{A}_p)(i)$ holds then X has the approximation property of type p for all $p \geq 1$.*

Proof. This follows from Proposition 2 and Proposition 4.8 in [7]. □

Theorem 4. *Let X be a Banach space having property $(\mathbb{A}_p), 1 \leq p \leq \infty$. Then $\mathcal{K}_p^d(X)$ is amenable.*

Proof. Let $\mathcal{U} = \mathcal{K}_p^d(X)\hat{\otimes}\mathcal{K}_p^d(X)$ and $\pi = \pi_{\mathcal{K}_p^d(X)}$. We will show that $\mathcal{K}_p^d(X)$ has an approximate diagonal in $\mathcal{U}$.

For $\alpha \in \mathcal{U}$, define $d_\alpha \in \mathcal{U}$ by

$$d_\alpha = \frac{1}{|G_\alpha|} \sum_{x \in G_\alpha} \Phi_\alpha(x) \otimes \Phi_\alpha(x^{-1}).$$

By condition $(\mathbb{A}_p)$(íí), $(d_\alpha : \alpha \in A)$ is a bounded net in $\mathcal{U}$. For,

$$\begin{aligned}\sum_{x \in G_\alpha} \|\Phi_\alpha(x)\|\|\Phi_\alpha(x^{-1})\| &\leq \sum_{x \in G_\alpha} \kappa_p^d(\Phi_\alpha(x))\kappa_p^d(\Phi_\alpha(x^{-1})) \\ &\leq (\sup\{\kappa_p^d(\Phi_\alpha(x) : x \in G_\alpha, \alpha \in A)\})^2 |G_\alpha|,\end{aligned}$$

by $(\mathbb{A}_p)$(íí), whence as an ideal norm, $\kappa_p^d(\cdot)$ leads to

$$\begin{aligned}\kappa_p^d(d_\alpha) &\leq \frac{1}{|G_\alpha|} \sum_{x \in G_\alpha} \|\Phi_\alpha(x)\|\|\Phi_\alpha(x^{-1})\| \\ &\leq (\sup\{\kappa_p^d(\Phi_\alpha(x) : x \in G_\alpha, \alpha \in A)\})^2.\end{aligned}$$

Take $T \in \mathcal{K}_p^d(X)$. By proposition 3, X has the approximation property of type p. Therefore $\mathcal{K}_p^d(X) = \overline{\mathcal{F}(X)}^{\mathcal{K}_p^d} = \Pi_p^{min}(X)$ by [7, Theorem 4.5] and [7, Remark 2.7], respectively. Since $\pi(d_\alpha) = P_\alpha$ by a straight forward computation, it follows from $(\mathbb{A}_p)$(i) that

$$\kappa_p^d\text{-}\lim_\alpha \pi(d_\alpha)T = T.$$

We also have

$$\kappa_p^d\text{-}\lim_\alpha P_\alpha T P_\alpha = T,$$

since

$$\begin{aligned}\kappa_p^d\text{-}\lim_\alpha (P_\alpha T P_\alpha - T) &= \kappa_p^d\text{-}\lim_\alpha - \kappa_p^d\text{-}\lim_\alpha P_\alpha T \\ &= \kappa_p^d\text{-}\lim_\alpha (P_\alpha(TP_\alpha - T)) \\ &= [\kappa_p^d\text{-}\lim_\alpha P_\alpha][\kappa_p^d\text{-}\lim_\alpha (TP_\alpha - T)] = 0.\end{aligned}$$

By Proposition 1, d_α is a diagonal of $\mathbb{M}_{n_\alpha}$ and so we have

$$P_\alpha T P_\alpha \cdot d_\alpha = d_\alpha \cdot P_\alpha T P_\alpha.$$

Thus

$$\kappa_p^d\text{-}\lim_\alpha (T \cdot d_\alpha - d_\alpha \cdot T) = \kappa_p^d\text{-}\lim_\alpha ((T - P_\alpha T P_\alpha)) \cdot d_\alpha - d_\alpha \cdot (T - P_\alpha T P_\alpha)) = 0.$$

This shows that $(d_\alpha : \alpha \in A)$ is an approximate diagonal for $\kappa_p^d(X)$. By Theorem 2, the Banach algebra $\kappa_p^d(X)$ is amenable. □

Corollary 1. *Let X be a Banach space having property $(\mathbb{A}_p), 1 \le p \le \infty$. Then $\Pi_p^{min}(X)$ is amenable.*

Proof. This follows from the proof of the previous theorem (Theorem (4)), Proposition 3, [7, Theorem 4.5] and [7, Remark 2.7]. □

The following lemma is inspired by the proof of [1, Theorem 4.1].

Lemma 2. *Let X be a Banach space, $\mathcal{F}$ be the operator ideal of all finite rank operators between Banach spaces and let γ be any operator ideal norm on $\mathcal{F}$. Suppose that (T_α) is a bounded approximate identity of bound λ for $\mathcal{A}(X)$. Then for every $F \in \mathcal{F}(X)$ it holds that $\lim_\alpha \gamma(F - T_\alpha F T_\alpha) = 0$.*

Proof. Write $F = RS$, with $R, S \in \mathcal{F}(X)$. Then

$$\begin{aligned}\gamma(F - T_\alpha F T_\alpha) &= \gamma((R - T_\alpha R)S + T_\alpha R(S - ST_\alpha)) \\ &\le \|R - T_\alpha R\|\gamma(S) + \lambda\gamma(R)\|S - ST_\alpha\|,\end{aligned}$$

and the right-hand side tends to zero as $\alpha \to \infty$. □

Theorem 5. *Let μ be a positive measure on a set S and let $p \in [1, \infty]$ and $q \in [1, \infty)$. Then the algebra $\mathcal{K}_p^d(L^q(\mu))$ is amenable.*

Proof. First consider the case $q > 1$ and let q' be the conjugate index to q. We will show that the condition $(\mathbb{A}_p)$(i) holds. With the notation as in [8], consider the collection of families $\mathcal{S}$ of finitely many, pairwise disjoint, measurable subsets L of $\mathcal{S}$ with $0 < \mu(L) < \infty$. Set $\mathcal{S}_1 \prec \mathcal{S}_2$ if each member of $\mathcal{S}_1$ is the union of a subfamily of $\mathcal{S}_2$. The biorthogonal system corresponding to $\mathcal{S} = \{L_1, ..., L_n\}$ is $((\chi_{L_1}/\mu(L_1)^{1/p}, ..., \chi_{L_n}/\mu(L_n)^{1/p}), (\chi_{L_1}/\mu(L_1)^{1/q}, ..., \chi_{L_n}/\mu(L_n)^{1/q}))$, the corresponding homomorphism into $\mathcal{F}(L^p(\mu))$ is denoted by $\Phi_{\mathcal{S}}$ with the corresponding projection denoted by $P_{\mathcal{S}}$. Note that

$$\| \sum_{i\leq n} \alpha_i \chi_{L_i}/\mu(L_i)^{1/q} \| = \sum_{i\leq n} |\alpha_i|^q \quad (\alpha_1, \ldots, \alpha_n \in \mathbb{C}).$$

Then for each $\mathcal{S} = \{L_1, \ldots, L_n\}$ we have $\|P_{\mathcal{S}}\| \leq 1$ since for every $g \in L^q(\mu)$ it follows that

$$\begin{aligned}
\|P_{\mathcal{S}(g)}\|_q^q &= \| \sum_{i\leq n} (\frac{\chi_{L_i}}{\mu(L_i)^{\frac{1}{q'}}} \otimes \frac{\chi_{L_i}}{\mu(L_i)^{\frac{1}{q}}})(g)\|_q^q \\
&= \| \sum_{i\leq n} \mu(L_i)^{-1/q'} (\int_{L_i} g) \mu(L_i)^{-1/q} \chi_{L_i} \|_q^q \\
&= \| \sum_{i\leq n} \frac{\chi_{L_i}}{\mu(L_i)^{1/q'}}(g) \frac{\chi_{L_i}}{\mu(L_i)^{\frac{1}{q}}} \|_q^q \\
&= \sum_{i\leq n} \mu(L_i)^{-q/q'} (|\int_{L_i} g|)^q \\
&\leq \sum_{i\leq n} \mu(L_i)^{-q/q'} \mu(L_i)^{q/q'} \int_{L_i} |g|^q \quad \text{[by Hölder's inequality]} \\
&= \sum_{i\leq n} \int_{L_i} |g|^q \\
&= \|g\|_q^q.
\end{aligned}$$

Furthermore, for each $L \in \mathcal{S}, \mathcal{P}_{\mathcal{S}}(\chi_L) = \mathcal{P}'_{\mathcal{S}}(\chi_L) = \chi_L$. For

$$\begin{aligned}
P_{\mathcal{S}}(\chi_L) &= \sum_{i\leq n} \frac{\chi_{L_i}}{\mu(L_i)^{\frac{1}{q'}}}(\chi_L) \frac{\chi_{L_i}}{\mu(L_i)^{\frac{1}{q}}} \\
&= \sum_{i\leq n} (\sum_{k\leq n} \int_{L_k} \frac{\chi_{L_i}(w)}{\mu(L_i)^{\frac{1}{q}}} \chi_L(w)) \frac{\chi_{L_i}}{\mu(L_i)^{\frac{1}{p}}} \\
&= \frac{1}{\mu(L)} (\int_L \chi_L(w)) \chi_L \\
&= \chi_L.
\end{aligned}$$

Similarly,

$$\begin{aligned}
P'_{\mathcal{S}}(\chi_L) &= \sum_{i\leq n} (\frac{\hat{\chi_{L_i}}}{\mu(L_i)^{\frac{1}{q}}})(\chi_L) \frac{\chi_{L_i}}{\mu(L_i)^{\frac{1}{q'}}} \\
&= \sum_{i\leq n} \frac{1}{\mu(L_i)} (\int \chi_L(w) \chi_{L_i}(w) d\mu) \chi_{L_i} \\
&= \frac{1}{\mu_(L)} \sum_{k\leq n} (\int_{L_k} \chi_L(w) d\mu) \chi_L \\
&= \frac{1}{\mu(L)} \int_L \chi_L(w) d\mu \chi_L \\
&= \chi_L.
\end{aligned}$$

For every $f \in L^q(\mu)$ there exists a sequence (f_n) of simple functions in $L^q(\mu)$ such that

$$\int |f_n(w) - f(w)|^q d\mu \longrightarrow 0,$$

whereas with $f_k = \sum_{i\leq n} \alpha_i{}^{(k)}\chi_{L_i}$, say, we have $P_{\mathcal{S}}f = f_k = \sum_{i\leq n} \alpha_i{}^{(k)}P_{\mathcal{S}}(\chi_{L_i}) = f_k$ from the above deliberations. It follows from $\|\cdot\|$-boundedness
$(= \|\cdot\|_{op}-$boundedness) of each $P_{\mathcal{S}}$ that

$$\begin{aligned} P_{\mathcal{S}}f &= \lim_k P_{\mathcal{S}}f_k \\ &= \lim_k f_k \\ &= f. \end{aligned} \tag{3.5}$$

Now fix $f \in L^q(\mu)$ and let U be any neighbourhood of f. Then, by Eqn (3.5), $P_{\mathcal{S}}f \in U$, whence it follows that there exists S_0 (namely, any S) such that $P_{\mathcal{S}}f \in U$ for every $\mathcal{S}$ such that $\mathcal{S}_0 \prec \mathcal{S}$. It follows that $P_{\mathcal{S}}f \longrightarrow f$ or that

$$\lim_{\mathcal{S}} P_{\mathcal{S}}f = f. \tag{3.6}$$

Similarly, for all $\lambda \in L^{q'}(\mu)$,

$$|P'_{\mathcal{S}}(\lambda)(f) - \lambda(f)| = |\lambda(P_{\mathcal{S}}f) - \lambda(f)| \leq \|\lambda\|\|P_{\mathcal{S}}(f)\| = 0.$$

Hence $P'_{\mathcal{S}}(\lambda)(f) - \lambda(f) = 0$ for all f and so

$$P'_{\mathcal{S}}(\lambda) = \lambda. \tag{3.7}$$

Whence $\lim_{\mathcal{S}} P'_{\mathcal{S}}(\lambda) = \lambda$ by a similar reasoning. Therefore $P_{\mathcal{S}}$ is a left approximate identity for $\mathcal{A}(L^q(\mu))$ and also a right approximate identity for $(L^q(\mu)$. Since $\|P_{\mathcal{S}}\| \leq 1$ for every $\mathcal{S}$, it follows from Lemma 2 that $\lim_{\mathcal{S}} \kappa_p^d(T - P_{\mathcal{S}}TP_{\mathcal{S}}) = 0$ for every $T \in \Pi_p^{min}(L_q(\mu))$; that is, $(P_{\mathcal{S}})$ is a bounded approximate identity in $\Pi_p^{min}(L_q(\mu))$.

By Eqn(3.4) we have that

$$\kappa_p^d(P_{\mathcal{S}}) \leq \|(\chi_{L_i}/\mu(L_i)^{1/q'})\|_{p'}^w \|(\chi_{L_i}/\mu(L_i)^{1/q})\|_p^w.$$

Since

$$(\sum_{i\leq n_{\mathcal{S}}} |\frac{\chi_{L_k}}{\mu(L_k)^{1/q}}(\frac{\chi_{L_i}}{\mu(L_i)^{1/q}})|^{p'})^{1/p'} = 1,$$

it follows that $\|(\chi_{L_i}/\mu(L_i)^{1/q'})\|_{p'}^w = 1$.
Similarly,

$$(\sum_{i\leq n_{\mathcal{S}}} |\frac{\chi_{L_i}}{\mu(L_k)^{1/q'}}(\frac{\chi_{L_i}}{\mu(L_i)^{1/q}})|^{p})^{1/p} = 1$$

implies that $\|(\chi_{L_i}/\mu(L_i)^{1/q})\|_p^w = 1$.
Therefore, $\kappa_p^d(P_{\mathcal{S}}) \leq 1$ for every $\mathcal{S}$, and hence $(P_{\mathcal{S}})$ is a bounded net in $\Pi_p^{min}(L^q(\mu))$. The bound of 1 in this case improves on the bound of $n_{\mathcal{S}}$ one would obtain if Eqn(3.3) were used, and which equation would hold via the deliberations using the left injective tensor norm $/d_p(\cdot)$ associated to $d_p(\cdot)$ as before, since $L^q(\mu)$ has the approximation property, and hence, has the κ_p- approximation property as well by [5, Proposition 3.10], and so, $\kappa_p^d(L^q(\mu)) \stackrel{1}{=} L^{q'}(\mu) \overset{/d_p}{\otimes} L^q(\mu)$ does hold by [5, Proposition 3.11]. To sum up, we have proved that $(P_{\mathcal{S}})$ is a bounded approximate identity for $\Pi_p^{min}(L^q(\mu))$ and so the condition $(\mathbb{A}_p)$(ı) holds.

Next let $\mathcal{S} = \{L_1, \ldots, L_{n_{\mathcal{S}}}\}$ and let $G_{\mathcal{S}}$ be the group of matrices of the form $D_{\mathbf{t}}E_\sigma$, where $D_{\mathbf{t}}$ is the diagonal matrix defined by $\mathbf{t} = (t_i\delta_{i,j})$, and where we have taken $t_i, \ldots, t_{n_{\mathcal{S}}} \in \{-1, 1\}$ and E_σ is the matrix which corresponds to permutation σ of $\mathbb{N}_{n_{\mathcal{S}}}$. Then $G_{\mathcal{S}}$ is an irreducible $n_{\mathcal{S}} \times n_{\mathcal{S}}$ matrix group. Observe that, if $x = D_{\mathbf{t}}E_{\sigma} \in G_{\mathcal{S}}$, then we have

$$\Phi_{\mathcal{S}}(x) = \sum_{i \le n_{\mathcal{S}}} t_i \mu(L_i)^{-1/q'} \chi_{L_i} \otimes \mu(L_{\sigma(i)})^{-1/q} \chi_{L_{\sigma(i)}}.$$

Hence

$$\kappa_p^d(\Phi_{\mathcal{S}}(x)) \le \|(t_i \chi_{L_i}/\mu(L_i)^{1/q'})\|_{p'}^w \|(\chi_{L_{\sigma(i)}}/\mu(L_{\sigma(i)})^{1/q'})\|_p^w$$

by Eqn(3.4). As before, $(\sum_{i \le n_{\mathcal{S}}} |\frac{\chi_{L_k}}{\mu(L_k)^{1/q}}(\frac{\chi_{L_i}}{\mu(L_i)^{1/q}})|^{p'})^{1/p'} = 1$, and so

$$\|(t_i \chi_{L_i}/\mu(L_i)^{1/q'})\|_{p'}^w = 1.$$

Similarly, $(\sum_{i \le n_{\mathcal{S}}} |\frac{\chi_{L_i}}{\mu(L_k)^{1/q'}}(\frac{\chi_{L_i}}{\mu(L_i)^{1/q}})|^p)^{1/p} = 1$, so that $\|(\chi_{L_{\sigma(i)}}/\mu(L_{\sigma(i)})^{1/q'})\|_p^w = 1$. Therefore, $\kappa_p^d(\Phi_{\mathcal{S}}(x)) \le 1$, for every $\mathcal{S}$ and for every $x \in G_{\mathcal{S}}$, and hence

$$\sup\{\kappa_p^d(\Phi_{\mathcal{S}}(x)) : x \in G_{\mathcal{S}}, \text{families } \mathcal{S}\} < \infty,$$

and this proves the condition $\mathbb{A}_p$(iί). Thus $L^q(\mu)$ has property $(\mathbb{A}_p)$, and the result follows in this case (where $q > 1$).

Now suppose that $q = 1$ and $\mu(\mathcal{S}) < \infty$. Then the above argument, with small notational changes shows that $L^1(\mu)$ has the symmetric property $(\mathbb{A}_p)$, and hence has an approximate diagonal of κ_p^d-bound 1 for $\mathcal{K}_p^d(L^1(\mu))$.

Finally consider the case $q = 1$ and μ a general positive measure, not necessarily σ-finite. For each measurable subset T of S, we regard $L^1(\mu|_T)$ as a closed linear subspace of $L^1(\mu)$. The approximate diagonals for $\mathcal{K}_p^d(L^1(\mu|_T))$ of $\kappa_p^d(\cdot)$-bound 1 constructed as above fit together in an obvious way to give another bounded net in $L^1(\mu)\hat{\otimes}L^1(\mu)$ such that this net is an approximate diagonal for $\mathcal{K}_p^d(L^1(\mu))$. Thus $\mathcal{K}_p^d(L^1(\mu))$ is amenable. $\square$

Then the classical case [8, Corollary 5.6.64] follows as a special case when $p = \infty$.

Corollary 2. *Let μ be a positive measure on a set $\mathcal{S}$ and let $q \in [1, \infty)$. Then the algebra $\mathcal{K}(L^q(\mu))$ is amenable.*

It will be pleasing to have a characterization of amenable Banach algebras of p-compact operators in terms of the symmetric property $\mathbb{A}_p$. Such hopes are dashed by the following observation:

Recall that

$$Property(\mathbb{A}_p) \Rightarrow Property(\mathbb{A}) \Rightarrow BAP \Rightarrow AP \Rightarrow AP_p, \forall\, p \ge 1.$$

The converse fails, as the next proposition asserts.

Proposition 4. *X has $AP_p \not\Rightarrow X$ has property $(\mathbb{A}_p)$ $(1 \le p < \infty)$.*

Proof. By [7, Example 4.9], there exists a subspace Y (say) of ℓ_q, $1 \le q < 2$ without AP, and so without property $(\mathbb{A})$ (else it would have BAP and hence AP) but which has AP_p for each $p \ge 2$. This space cannot have property $(\mathbb{A}_p)$ either; else it would have property $(\mathbb{A})$. $\square$

Theorem 6. *Let X be a Banach space. If X' has property $(\mathbb{A}_p)$, then X has property $(\mathbb{A}_p)$.*

Proof. Suppose X^* has property $(\mathbb{A}_p)$. Then there exists a net

$$((x_{i,\alpha}^*)_{i \le n_\alpha}, (x_{i,\alpha}^{**})_{i \le n_\alpha}) \quad (\alpha \in A),$$

of biorthogonal systems of size n_α in $\ell_p^s(X') \times \ell_{p'}^w(X'')$ that satisfies the requirements of property $(\mathbb{A}_p)$ for X'.

Next, we construct a net of finite biorthogonal systems for X as follows:

Let $\mathcal{U}$ and $\mathcal{V}$ be the sets of all finite dimensional subspaces of X and X' respectively. Fix $U \in \mathcal{U}$ and $V \in \mathcal{V}$. The principle of local reflexivity guarantees the existence of a linear map (hence choose it)

$$S_{U,V,\alpha} : \mathrm{span}(\{x_{i,\alpha}^{**} | i = 1, \dots, n_\alpha\} \cup U) \longrightarrow X$$

satisfying

(1) $\|S_{U,V,\alpha}\| \leq 2$,

(2) $S_{U,V,\alpha}|_U = 1_U$,

(3) $\langle S_{U,V,\alpha}(x_{i,\alpha}^{**}), x^* \rangle \ = \ \langle x^*, x_{i,\alpha}^{**} \rangle$ for all $x^* \in \mathrm{span}(\{x_{i,\alpha}^*\} \cup V)$.

Order $\mathcal{U} \times \mathcal{V} \times A$ by the cartesian product order and $\mathcal{U}$, $\mathcal{V}$ by containment. By our construction,

$$\{(S_{U,V,\alpha} x_{i,\alpha}^{**}, x_{j,\alpha}^*) | i, j = 1, \dots, n_\alpha\} \quad ((U, V, \alpha) \in \mathcal{U} \times \mathcal{V} \times A)$$

is the desired net of finite biorthogonal systems.

We denote the lifts of matrix algebras and the projections that are associated to $S_{U,V,\alpha}$ by $\Phi_{U,V,\alpha}$ and $P_{U,V,\alpha}$, respectively. Thus, we have

$$\Phi_{U,V,\alpha} = S_{U,V,\alpha} \Phi'_\alpha i_X \ \text{ and } \ P_{U,V,\alpha} = S_{U,V,\alpha} P'_\alpha i_X,$$

where i_X is the natural inclusion of X into X'', and P_α's are the property $(\mathbb{A}_p)$ projections for X'. Then $\{P_{U,V,\alpha}\}$ is a $\kappa_p^d(\cdot)$-bounded set since

$$\begin{aligned}
\kappa_p^d(P_{U,V,\alpha}) &= \kappa_p^d(S_{U,V,\alpha} P'_\alpha i_X) \\
&\leq \|S_{U,V,\alpha}\| \kappa_p^d(P'_\alpha) \|i_X\| \\
&\leq 2\kappa_p^d(P'_\alpha), \text{(by} \quad (1) \quad \text{above)} \\
&= 2\kappa_p(P''_\alpha) \\
&= 2 \sum_{i \leq n_\alpha} \|x_{i,\alpha}^*\| \|x_{i,\alpha}^{**}\| \\
&= 2 \|(x_{i,\alpha}^{**})\|_{p'}^w \|(x_{i,\alpha}^*)\|_p^s
\end{aligned}$$

by Eqn(3.2). Hence, the $P_{U,V,\alpha}$'s belong to $\Pi_p^{min}(X)$ (alternatively, this follows from the fact that the $P_{U,V,\alpha}$'s are finite rank operators on X and [7, Remark 2.7] applies).

Next we show that $\kappa_p^d(P_{U,V,\alpha})$ is uniformly bounded in (U, V, α), and so, $(\mathbb{A}_p)$(íí) holds for X.

Each $P_\alpha : X' \longrightarrow X'$ is weakly compact (as a compact operator). Let $R := P'_\alpha|_X$, where $P'_\alpha|_X : X \longrightarrow X''$. Then $R'' = P'_\alpha$ (since R is also a weakly compact operator) by Lemma 1. Since the P_α's are the Property $\mathbb{A}_p$ projections for X', it follows that $\kappa_p^d(P_\alpha)$ is uniformly bounded by $(\mathbb{A}_p)$(íí) and, hence, so is $\kappa_p(P'_\alpha)$ by definition. Therefore, $\kappa_p(P'_{U,V,\alpha}) = \kappa_p(R'') = \kappa_p(P'_\alpha|_X) \leq \kappa_p(P'_\alpha) < \infty$ in (U, V, α). Hence, $\kappa_p^d(P_{U,V,\alpha}) < \infty$ in (U, V, α). That is $(P_{U,V,\alpha})_{U,V,\alpha}$ is a bounded net in $\Pi_p^{min}(X)$. Our last mission is to show that the net $(P_{U,V,\alpha})_{U,V,\alpha}$ is an approximate identity in $\Pi_p^{min}(X)$.

Since property $(\mathbb{A}_p)$ for X' implies property $(\mathbb{A})$ for X', it follows that P_α's are the property $(\mathbb{A})$ projections for X', a net of finite biorthogonal system is found as in [11, Theorem 4.3], namely $(P_{U,V,\lambda})$. Thus $(P_{U,V,\lambda})_{U,V,\alpha}$ is a $(\kappa_p^d(\cdot))$-bounded) approximate identity for $\mathcal{A}(X)$. It follows from Lemma 2, that for every $F \in \mathcal{F}(X)$ it holds that $\lim_{U,V,\lambda} \kappa_p^d(F - P_{U,V,\lambda} F P_{U,V,\lambda}) = 0$.

Now, let $T \in \Pi_p^{min}(X) = \overline{\mathcal{F}(X)}^{\kappa_p^d(\cdot)}$ be an arbitrarily chosen operator. Then there exists a net $(T_\alpha) \subset \mathcal{F}(X)$ such that $T = \kappa_p^d - \lim T_\alpha$. Hence

$$\lim_{U,V,\lambda} \kappa_p^d(T_\alpha - P_{U,V,\lambda} T P_{U,V,\lambda}) = 0.$$

Write

$$\begin{aligned} T - P_{U,V,\lambda} T P_{U,V,\lambda} &= (T - T_\alpha) + (T_\alpha - P_{U,V,\lambda} T_\alpha P_{U,V,\lambda}) \\ &+ (P_{U,V,\lambda} T_\alpha P_{U,V,\lambda} - P_{U,V,\lambda} T P_{U,V,\lambda}) \\ &= (T - T_\alpha) + (T_\alpha - P_{U,V,\lambda} T_\alpha P_{U,V,\lambda}) \\ &+ P_{U,V,\lambda}(T_\alpha - T) P_{U,V,\lambda}. \end{aligned}$$

Then

$$\begin{aligned} \kappa_p^d(T - P_{U,V,\lambda} T P_{U,V,\lambda}) &\le \kappa_p^d(T - T_\alpha) + \kappa_p^d(T_\alpha - P_{U,V,\lambda} T_\alpha P_{U,V,\lambda}) \\ &+ \kappa_p[P_{U,V,\lambda}(T_\alpha - T) P_{U,V,\lambda}] \\ &\le \kappa_p^d(T - T_\alpha) + \kappa_p^d(T_\alpha - P_{U,V,\lambda} T_\alpha P_{U,V,\lambda}) \\ &+ \|P_{U,V,\lambda}\| \kappa_p^d(T_\alpha - T) \|P_{U,V,\lambda}\|. \end{aligned}$$

Hence

$$\lim_{U,V,\lambda} \kappa_p^d(T - P_{U,V,\lambda} T P_{U,V,\lambda}) = 0 \tag{3.8}$$

for every $T \in \Pi_p^{min}(X)$. From Eqn(3.8) we have

$$\begin{aligned} \kappa_p^d(T - P_{U,V,\lambda} T) &= \kappa_p^d[T - P_{U,V,\lambda} T P_{U,V,\lambda} + P_{U,V,\lambda}(T P_{U,V,\lambda} - T)] \\ &\le \kappa_p^d[T - P_{U,V,\lambda} T P_{U,V,\lambda}] + \kappa_p^d[P_{U,V,\lambda}] \|T P_{U,V,\lambda} - T\| \\ &\to 0 \end{aligned}$$

using the fact that $(P_{U,V,\lambda})$ is a $\|\cdot\|$-bounded approximate identity for $\mathcal{A}(X)$ by [11, Theorem 4.3], as alluded to above.

Similarly,

$$\begin{aligned} \kappa_p^d(T - T P_{U,V,\lambda}) &= \kappa_p^d[T - P_{U,V,\lambda} T P_{U,V,\lambda} + (P_{U,V,\lambda} T - T) P_{U,V,\lambda}] \\ &\le \kappa_p^d[T - P_{U,V,\lambda} T P_{U,V,\lambda}] + \|P_{U,V,\lambda} T - T\| \kappa_p^d(P_{U,V,\lambda}) \\ &\to 0. \end{aligned}$$

This completes the proof that $(P_{U,V,\lambda})_{U,V,\alpha}$ is a ($\kappa_p^d(\cdot)$-bounded) approximate identity for $\Pi_p^{min}(X)$. □

Corollary 3. *The dual of the Enflo space does not have property* $(\mathbb{A}_p)$.

Proof. If the dual of the Enflo space had property $(\mathbb{A}_p)$, then the Enflo space itself would have the approximation property. □

Theorem 7. *Let K be a compact Hausdorff space and let (Ω, Σ, μ) be a measure space. Then $\mathcal{K}_p^d(C(K))$ and $\mathcal{K}_p^d(L^\infty(\mu))$, $p \in [1, \infty)$, are amenable.*

Proof. It was shown in the proof of Theorem 5 that $L^q(\mu)$ has property $\mathbb{A}_p$ whenever $1 \le q < \infty$. In particular, $L^1(\mu)$ has property $(\mathbb{A}_p)$.

Since $C(K)' = L^1(\mu_K)$ for some measure space $(\Omega_K, \Sigma_K, \mu_k)$, and $L^\infty(\mu) = C(K_\mu)$ for a well-chosen compact space K_μ, Theorem 6 above implies that $C(K)$ and $L^\infty(\mu)$ have property $\mathbb{A}_p$ as well. The proof is concluded by appealing to Theorem 4. □

4. Open Question and Concluding Remarks

By Remark 2, also Definition 1, we see that property $(\mathbb{A}_p)$ is not implied by property $(\mathbb{A})$ of Grønbæk, Johnson and Willis [11, Theorem 4.2]. We seek a concrete Banach space which establishes nonequivalence of these two geometric properties:

Question 8. *Give an example of a Banach space X such that X has property $(\mathbb{A})$ but lacks property $(\mathbb{A}_p)$.*

In this paper, we studied the notion of amenability within the framework of Banach algebras of p-compact operators and its closely related ideals. We investigated this notion by means of the geometric property $(\mathbb{A}_p)$. It is a rather strong condition on the underlying Banach space. However, interesting observations and consequences like Proposition 4 and Theorem 4, among others, exhibit it as a noteworthy geometric property of study in connection with amenability of Banach operator algebras at the p-level.

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