An Asymptotic Analysis of Default-Free Zero-Coupon Bond Pricing in Single-Factor Models

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Abstract: Asymptotic analysis is a powerful technique that has been used with success to obtain approximate solutions to complicated problems in applied mathematics. In this paper, this concept is exploited to obtain new insights into a problem arising in mathematical finance. We are concerned with solutions of the bond pricing equation in single-factor models. In particular, we obtain an approximate generalised asymptotic solution (which amalgamates currently two major approaches to interest rate modelling) to the problem of "long-term-to-maturity" zero-coupon bond pricing; we also consider two popular models in the literature to demonstrate how this generalised solution may be employed in practice. Furthermore, we study the problem of "short-term-to-maturity" zero-coupon bond pricing where the volatility, contrary to a number of conventional approaches in the literature, is obtained as part of the solution to the problem; the results are curious and fascinating.

1 Introduction

Yield curve modelling, including the pricing and hedging of both interest rate derivatives and fixed income products, has become an important area of research in mathematical finance. This is partly as a result of an increasingly wide range of traded interest rate dependent products in the market, partly because the interest rate derivatives market is the largest of all derivative markets, and partly because of the complexity of the issues surrounding interest rate modelling. One frequently adopted technique of yield curve modelling assumes that the component interest rates are driven by a number of common random variables, which in turn are governed by stochastic differential equations.

The assumption of an affine drift and squared diffusion process leads to the class of models described in Duffie (1996) and Duffie & Kan (1996). This class includes most of the commonly used one-factor interest rate models (e.g., Ho & Lee, 1986, Vasicek, 1977 and Cox *et al.*, 1985). The simplest

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assumption within this modelling technique is to require that there is only one source of randomness. In this case it is supposed that at any time t an interest rate r (often termed the instantaneous spot-rate or the short-term interest rate) is governed by a stochastic differential equation of the form:

$$dr(t) = u(r,t)dt + w(r,t)dX. \tag{1}$$

The deterministic functions $u(r,t)$ and $w(r,t)$ dictate the behaviour of r, and X is a standard Brownian motion (i.e. a continuous-time process with increments that are normally distributed under the real-world measure). With this stochastic model for $r(t)$, all other interest rates (i.e. the instantaneous yield curve) are recovered through the pricing of a series of zero-coupon bonds $V(r,t;T_i)$ with arbitrary maturities T_i. Without loss of generality, a solution for $V(r,t;T_i)$ may be obtained by considering the equation:

$$\frac{\partial V}{\partial t} + \frac{w^2}{2}\frac{\partial^2 V}{\partial r^2} - a(r,t)\frac{\partial V}{\partial r} - rV = 0, \tag{2}$$

for some (as yet unknown) function $a(r,t)$, and for arbitrary maturity T.

Although the single-factor approach is generally perceived to be somewhat restrictive, and even naive, in comparison to more recent developments in interest rate modelling (see for example, Heath *et al.*, 1992 or Brace *et al.*, 1997) it is both historically interesting and functional in many markets. The reason for this functionality is the relative complexity of more involved models and their increased requirements in terms of yield curve liquidity and dynamics. Emerging markets, in particular, suffer from yield curve constraints which make more involved models impractical. From our point of view, however, we are keen to develop this methodology in the simplest of scenarios, before attempting to extend it to multi-factor models (affine or otherwise.)

2 Boundary Conditions

In order to solve (2) uniquely for V, we need to prescribe one temporal and two spatial boundary conditions. The temporal boundary condition is standard. At maturity, $t = T$, the value of the bond $V(r,t;T)$ is known precisely, and is given by:

$$V(r,T;T) = V_T. \tag{3}$$

The other boundary conditions for this problem are not obvious at all and therefore warrant some discussion.

Traditionally, the spatial boundary conditions are set at $r = 0$ and at $r = \infty$. But in reality, these values are unattainable: Under reasonable economic conditions, interest rates take values in a fairly narrow band, and in most instances the reference interest rate is set by a central authority. In cases of extreme interest rate movements, there are more serious implications to the economy than the maintenance of the arbitrage-free value of zero-coupon bonds.

However, since (as it will become evident later) the problem of short-term-to-maturity bond pricing is dominated by the volatility term, $w(r,t)$, in equation (1), we will then follow tradition in this case. But, for the problem of long-term-to-maturity bond pricing (which – as it will be seen later – is dominated by the drift term, $u(r,t)$, in equation (1)), we wish to restrict attention to the region between two values for r, r_{min} and r_{max}. (That is to say, we assume that the short-term interest rate is confined to lie between two, known values r_{min} and r_{max}. These extrema may be viewed as absorbtion barriers beyond which the interest rate values are undetermined.) In so doing, we incorporate the ideas and approach of Lewicki & Avellaneda (1996), and Epstein & Wilmott (1998). This requires a reworking of the boundary conditions. Instead of defining a value for the solution of (2) at some extremes, we will marry the boun! dary conditions to the solution of the problem proposed by Epstein & Wilmott (1998).

In this approach, for any evolution of the short-term interest rate r (in particular, not necessarily Brownian), the value of the bond $V(r,t)$ always lies between two *functions* $V_{min}(r,t)$ and $V_{max}(r,t)$. The functions $V_{min}(r,t)$ and $V_{max}(r,t)$ are, respectively, values of the zero-coupon bond in the worst-case and best-case scenarios at each time t and are determined by examining the random evolution of the interest rate. *They are not arrived at by solving equation (2)*; and for our analysis here, we consider them as known or given. We will seek a solution to (2) where, if the risk-free rate of interest is approaching its maximum value (note that, within reasonable models, expected values of interest rates should not tend to infinity, see Rebonato (1996) for a discussion of this condition), then the value of the bond will be nearing its minimum. Therefore we can expect that at $r = r_{max}$, $V(r,t;T)$ and $V_{min}(r,t;T)$ coincide:

$$V(r_{max}, t; T) = V_{min}(r_{max}, t; T), \tag{4}$$

where r_{max} is the upper absorbtion barrier of r. We will assume further that r_{max} is static and known. In fact, as will be seen later in Section 4, in the relevant case of a long-term-to-maturity bond, r_{max} and r_{min} should, in general, be determined from a suitable construction of $u(r,t)$ in equation (1).

Similarly, $V(r,t;T)$ will approach the function $V_{max}(r,t;T)$ as r approaches r_{min}:

$$V(r_{min}, t; T) = V_{max}(r_{min}, t; T). \tag{5}$$

Again, in general, the functions $V_{max}(r,t;T)$ and $V_{min}(r,t;T)$ have to be obtained by solving an appropriate problem; but here we assume, for simplicity and briefty, that they are already known or given.

It should be mentioned at this point that the idea of constant and known extreme values for the instantaneous spot-rate, r_{min} and r_{max}, is highly arbitrary and artificial. In general, one should model r_{min} and r_{max} as functions of t.

3 Asymptotics

We wish to perform an asymptotic analysis of equation (2). (A number of papers in the literature have exploited asymptotic methods to examine financial problems, e.g. Atkinson & Wilmott (1995) on portfolio management, and Whalley & Wilmott (1997) on optimal hedging.) This is in contrast to what is usually meant by an asymptotic analysis in finance, in which the solution of the differential equation is examined near the boundaries. We are interested in dominance issues. Equation (2) is dominated by different terms depending on the value of T, the maturity of the zero-coupon bond we are solving for. This leads to markedly different behaviour of the differential equation in the separate cases of short-term-to-maturity (STTM) and long-term-to-maturity (LTTM) bonds. Consequently, we are able to use the simple framework of the single-factor model to generate different solutions and different behaviours for the short and long ends of the yield curve. In part, this eases the major drawback of single-factor models, namely the serial correlation of the derived yields for all maturities. Not surprisingly, we find that the solution to equation (2) for STTM bonds is dominated by the volatility term in equation (1), $w(r,t)$, and that the corresponding solution for LTTM bonds is dominated by the drift term, $u(r,t)$. This is *not* the major insight of this analysis, however.

In order to compare terms of similar order, we need to define non-dimensional variables in (2). Initially, we scale the variables by setting:

$$\begin{aligned} V &= V_T\bar{V}, \\ t &= T\bar{t}, \\ r &= R\bar{r}, \end{aligned}$$

where T is the maturity of the zero-coupon bond, and R is a "typical" value of the instantaneous spot-rate (in some models, this can be viewed as the mean-reversion level of r). V_T is the known maturity (par) value of $V(r,t;T)$. The "barred" variables are now non-dimensional. Furthermore, we have that (see for example, Wilmott, 1998):

$$dX \sim \sqrt{dt}.$$

Consequently, if we scale the volatility w in equation (1), with:

$$w = \frac{1}{T^{\frac{3}{2}}}\bar{w};$$

then $\bar{w}$ is dimensionless. Using the above scalings in equation (2) we obtain a non-dimensional partial differential equation governing the evolution of $\bar{V}$:

$$\frac{1}{RT}\frac{\partial \bar{V}}{\partial \bar{t}} + \frac{1}{2}\frac{1}{(RT)^3}\bar{w}^2\frac{\partial^2 \bar{V}}{\partial \bar{r}^2} - \bar{a}\frac{\partial \bar{V}}{\partial \bar{r}} - \bar{r}\bar{V} = 0, \tag{6}$$

where it has been instructive to scale a such that $a = R^2\bar{a}$.

In order to complete the non-dimensionalisation of this problem, we scale the dimensional variables $u(r,t)$ and X in equation (1) such that:

$$\begin{aligned} u &= \frac{R}{T}\bar{u}, \\ X &= T^{\frac{1}{2}}\bar{X}, \end{aligned}$$

where, as before, the barred variables are now non-dimensional. These scalings are consistent with the scalings performed on the other variables t, r, and w. Under these conditions, equation (1) becomes:

$$d\bar{r}(\bar{t}) = \bar{u}(\bar{r},\bar{t})d\bar{t} + \frac{1}{RT}\bar{w}(\bar{r},\bar{t})d\bar{X}, \tag{7}$$

which is non-dimensional.

We study this problem under two cases of interest, one of long-term-to-maturity (large (RT)) and the other of short-term-to-maturity (small (RT)).

4 Long-Term-to-Maturity Bond

If we take the typical value for the instantaneous short-rate to be of the order $R \sim 12\%$ per annum (a not untypical value in emerging markets), then for a bond of typical time to maturity $T \sim 20$ years, the non-dimensional parameter $1/RT \sim 0.42$ and $(1/RT)^3 \sim 0.07$ (which can be taken as "small"). We henceforth write $(1/RT)^3 = \epsilon$ $(\epsilon > 0)$. Equations (6) and (7), respectively, become:

$$\begin{aligned} \epsilon^{\frac{1}{3}}\frac{\partial V}{\partial t} + \frac{\epsilon}{2}w^2\frac{\partial^2 V}{\partial r^2} - a\frac{\partial V}{\partial r} - rV &= 0, && (8) \\ u(r,t)dt + \epsilon^{\frac{1}{3}}w(r,t)dX &= dr, && (9) \end{aligned}$$

where the bars have been dropped for convenience. We immediately observe from (9) that to leading order (ϵ^0), the instantaneous spot-rate is deterministic:

$$\frac{dr}{dt} = u(r,t). \tag{10}$$

However, as will be seen, the stochastic term cannot be ignored even if equation (8) were to be solved to lowest order only.

The non-dimensional boundary conditions for equation (8) transform to:

$$V(r,1) = 1,$$

$$\begin{aligned} V(r_M,t) &= V_m(r_M,t) \text{ where } r_M = \frac{r_{max}}{R} \text{ and } V_m = \frac{V_{min}}{V_T}, \forall\, t : 0 \le t \le T \\ V(r_m,t) &= V_M(r_m,t) \text{ where } r_m = \frac{r_{min}}{R} \text{ and } V_M = \frac{V_{max}}{V_T}, \forall\, t : 0 \le t \le T. \end{aligned}$$

Also, to lowest order, equation (8) is quasi-steady-state, i.e. we have to solve the steady-state problem at different times until maturity. This simply means, as one would expect, that to leading order, $V(r,1) = V_m(r,1) = V_M(r,1) = 1$.

4.1 Asymptotic Analysis

The smallness parameter ϵ appears in the coefficient of the highest order derivative term in equation (8). This implies that this is a singular perturbation problem (see for example, Hinch (1991) for an extensive discussion of this type of problem). The problem may be solved by following a standard expansion technique through seeking an outer expansion of V, V^o, of the form:

$$V^o(r,t;\epsilon) = V_0^o + \epsilon^{\frac{1}{3}} V_1^o + \epsilon^{\frac{2}{3}} V_2^o + \epsilon V_3^o + \cdots. \tag{11}$$

Substitution of (11) into (8), and comparison of coefficients of ϵ yields (truncated at $O(\epsilon^1)$):

$$\begin{aligned} a\frac{\partial V_0^o}{\partial r} + rV_0^o &= 0, \\ a\frac{\partial V_1^o}{\partial r} + rV_1^o &= \frac{\partial V_0^o}{\partial t}, \\ a\frac{\partial V_2^o}{\partial r} + rV_2^o &= \frac{\partial V_1^o}{\partial t}, \\ a\frac{\partial V_3^o}{\partial r} + rV_3^o &= \frac{\partial V_2^o}{\partial t} + \frac{w^2}{2}\frac{\partial^2 V_0^o}{\partial r^2}. \end{aligned}$$

These equations may be solved subject to the boundary condition:

$$V^o(r_m,t;\epsilon) = V_M(r_m,t) + \epsilon^{\frac{1}{3}}(0) + \epsilon^{\frac{2}{3}}(0) + \epsilon(0) + \cdots,$$

to give:

$$\begin{aligned} V^o(r,t;\epsilon) &= V_M(r_m,t) e^{-\int_0^r \frac{\nu(t)}{a(\nu,t)} d\nu} \left\{ 1 - \epsilon^{\frac{1}{3}} \int_0^r \frac{\nu(t)}{a^2(\nu,t)} \frac{d\nu}{dt} d\nu - \epsilon^{\frac{2}{3}} \left(\int_0^r \left\{ \frac{\nu(t)}{a^3(\nu,t)} \left(\frac{d\nu}{dt}\right)^2 \right.\right.\right. \\ &- \left.\left.\left. \frac{\nu(t)}{a^2(\nu,t)} \frac{d\nu}{dt} \int_0^\nu \frac{\mu(t)}{a^2(\mu,t)} \frac{d\mu}{dt} d\mu \right\} d\nu \right) \right\} + \cdots. \end{aligned} \tag{12}$$

The outer solution, equation (12), will then have to be matched with the inner solution of V in the assumed boundary layer, near $r = r_M$. (It should be noted that the existence of the boundary layer near this point is verified if the resulting expansions are matchable and the results are mathematically consistent (Nayfeh, 1994)). To obtain the inner solution, V^i, in the boundary layer, a stretched variable:

$$\xi = \frac{r_M - r}{\epsilon^n}, \text{ for } n > 0, \tag{13}$$

should be considered, where n is to be found as part of the analysis. (We will comment in detail about the implications of (13) with regard to equation (9) in due course.)

Rewriting equation (8), in terms of the stretched variable in (13), gives:

$$\epsilon^{\frac{1}{3}+n}\frac{\partial V^i}{\partial t} + \frac{\epsilon^{1-n}}{2}w^2\frac{\partial^2 V^i}{\partial \xi^2} + a\frac{\partial V^i}{\partial \xi} - \epsilon^n\left(r_M - \epsilon^n \xi\right)V^i = 0,$$

whose dominant part, as $\epsilon \to 0$, is:

$$\frac{\epsilon^{1-n}}{2}w^2\frac{\partial^2 V^i}{\partial \xi^2} + a\frac{\partial V^i}{\partial \xi} = 0. \tag{14}$$

The distinguished limit of (13), as $\epsilon \to 0$, is:

$$\frac{w^2}{2}\frac{\partial^2 V^i}{\partial \xi^2} + a\frac{\partial V^i}{\partial \xi} = 0, \tag{15}$$

iff $n = 1$. (It is important to note that if the boundary layer is assumed to be near the point $r = r_m$, the same analysis shows that $n = 1$, and equation (15) is slightly modified. We will exploit this result effectively later. Furthermore, there is nothing in the analysis near the point $r = r_m$, to preclude negative interest rates. This has obvious implications for the choice of model.) Equation (15) may then be solved by seeking a similar expansion to (11):

$$V^i(\xi, t; \epsilon) = V_0^i(\xi, t) + \epsilon^{\frac{1}{3}}V_1^i(\xi, t) + \epsilon^{\frac{2}{3}}V_2^i(\xi, t) + \epsilon V_3^i(\xi, t) + \cdots, \tag{16}$$

subject to the boundary condition:

$$V^i(0, t; \epsilon) = V_m(0, t) + \epsilon^{\frac{1}{3}}(0) + \epsilon^{\frac{2}{3}}(0) + \epsilon(0) + \cdots.$$

The remaining arbitrary functions of t, for each V_j^i $(j = 0, 1, 2, 3, \cdots)$, can then be found by matching (16) and (11).

We will, however, not adopt the above-outlined standard technique to solve this problem. There is a number of reasons for this. First of all, the matching process is not trivial at all, especially if it has to be performed numerically. Furthermore, in practice the problem will not be considered fully solved until appropriate functional forms of dr/dt (given by equation (10)), in addition to $a(r, t)$ and $w(r, t)$ have been proposed and motivated. In this regard, the literature is not unanimous, and different practitioners, researchers, and academics have suggested different forms of $u(r, t)$, $w(r, t)$, and $a(r, t)$. From a mathematical point of view, it is desirable to develop a general solution where neither $u(r, t)$, $w(r, t)$, nor $a(r, t)$ has been restricted to any particular functional form.

However, if one is to follow the standard method of solving this problem, it is clear from equation (12) that some specific functional forms of $u(r, t)$ and $a(r, t)$ (or even $w(r, t)$) are required before the matching process can be performed. It is only through this process that the location of the boundary layer can be verified. This also implies that the process must be repeated for each form of the functions $u(r, t)$, $w(r, t)$, and $a(r, t)$. Clearly, this process may be very tedious and time-consuming.

If the analysis and methodology are to be of practical benefit, a kind of "cheese-cutter" solution approach will be preferable, whereby a minimum of mathematics (if any) is required for use. Thus, it is desirable to develop a solution where model choices can be made with relative ease. In this case, it seems appropriate to resort to the method of multiple scales.

¿From the above analysis, we observe that V depends not only on the independent variables r, t and the smallness parameter ϵ, but is also determined by combinations of these (see for example r/ϵ in equation (13)). Consequently, this problem is suitable for treatment by the method of multiple scales. Furthermore, because the domain of r is finite, we do not have to worry about secular terms ϵr, $\epsilon^2 r$, $\epsilon^3 r$, $\cdots$ in the limit as $r \to \infty$. We introduce two variables which cater for the inclusion of an inner and outer solution simultaneously:

$$r_0 = r, \tag{17}$$

$$\xi = \frac{r_M - r}{\epsilon}. \tag{18}$$

The transformation (18) may be modified symmetrically for the existence of the boundary layer at either absorbtion barrier. Without loss of generality, we will use (18) in order to develop a general approximate solution for $V(r,t)$. The symmetry condition allows the analysis to be carried out at either extreme.

Now, equations (17) and (18) imply:

$$\begin{aligned} \frac{\partial}{\partial r} &= -\frac{1}{\epsilon}\frac{\partial}{\partial \xi} + \frac{\partial}{\partial r_0}, \\ \frac{\partial}{\partial r^2} &= \frac{1}{\epsilon^2}\frac{\partial}{\partial \xi^2} - \frac{2}{\epsilon}\frac{\partial}{\partial \xi \partial r_0} + \frac{\partial}{\partial r_0^2}. \end{aligned}$$

We are dealing with a set of coupled equations in this problem (the evolution of the derivative and the evolution of the state variable, r). Consequently, a re-scaling of r in (8) must be accompanied by an appropriate re-scaling of t, and hence of X, in (9). This is to ensure that the problem remains unchanged. We observe that, in this case, it is suitable to introduce two time-scales:

$$\begin{aligned} t &= t_0, \\ t_1 &= \frac{t}{\epsilon}, \end{aligned}$$

with the result that we have to set:

$$\begin{aligned} X &= X_0, \\ X_1 &= \frac{X}{\epsilon}. \end{aligned}$$

Then, to lowest order, equation (9) becomes:

$$dr_0 = u(\xi, r_0, t_1, t_0)dt_0 + \epsilon^{\frac{1}{3}} w(\xi, r_0, t_1, t_0)dX_0.$$

It is intuitive to think of this particular problem in the following fashion. There are two "types" of behaviours that are typical, one "short-term" associated with one time-scale, and one "long-term" associated with another, altogether different, time-scale.

A substitution of all the above transformations into (8) yields the following partial differential equation governing the behaviour of V throughout the ranges of the independent variables:

$$\epsilon^2 \frac{w^2}{2}\frac{\partial^2 V}{\partial r_0^2} + \epsilon^{\frac{4}{3}}\frac{\partial V}{\partial t_0} + \epsilon^{\frac{1}{3}}\frac{\partial V}{\partial t_1} - \epsilon\left(w^2\frac{\partial^2 V}{\partial\xi\partial r_0} + a\frac{\partial V}{\partial r_0} + r_0 V\right) + \frac{w^2}{2}\frac{\partial^2 V}{\partial \xi^2} + a\frac{\partial V}{\partial \xi} = 0. \quad (19)$$

We then seek a solution through a uniform expansion of V of the form:

$$V(r,t;\epsilon) = V_0(\xi,r_0,t_1,t_0) + \epsilon^{\frac{1}{3}}V_1(\xi,r_0,t_1,t_0) + \epsilon^{\frac{2}{3}}V_2(\xi,r_0,t_1,t_0) + \epsilon V_3(\xi,r_0,t_1,t_0) + \cdots. \quad (20)$$

Substituting (20) into (19) and comparing coefficients of ϵ we obtain the following $O(\epsilon^0)$ and $O(\epsilon^{\frac{1}{3}})$ equations respectively:

$$\frac{\partial^2 V_0}{\partial \xi^2} + \frac{2a}{w^2}\frac{\partial V_0}{\partial \xi} = 0, \quad (21)$$

$$\frac{\partial^2 V_1}{\partial \xi^2} + \frac{2a}{w^2}\frac{\partial V_1}{\partial \xi} = -\frac{2}{w^2}\frac{\partial V_0}{\partial t_1}. \quad (22)$$

The general solution of (21) is:

$$V_0(\xi,r_0,t_1,t_0) = A(r_0,t_1,t_0)\int_0^\xi e^{-\int_0^\nu \frac{2a}{w^2}d\mu}d\nu + B(r_0,t_1,t_0), \quad (23)$$

where $A(r_0,t_1,t_0)$ and $B(r_0,t_1,t_0)$ are determined by imposing a solvability condition on (22). Substitution of V_0 into equation (22) gives:

$$\frac{\partial^2 V_1}{\partial \xi^2} + \frac{2a}{w^2}\frac{\partial V_1}{\partial \xi} = -\frac{2}{w^2}\left\{\left(\frac{\partial A}{\partial t_1}\right)\int_0^\xi e^{-\int_0^\nu \frac{2a}{w^2}d\mu}d\nu + A\frac{\partial}{\partial t_1}\left(\int_0^\xi e^{-\int_0^\nu \frac{2a}{w^2}d\mu}d\nu\right) + \frac{\partial B}{\partial t_1}\right\}. \quad (24)$$

Ideally, we do not have to solve (24) but only need to demand that its solution be uniform, i.e. $\epsilon^{\frac{1}{3}}V_1$ does not grow faster than V_0 as $\xi \to \infty$.

Effectively, by obtaining the appropriate form of A and B in equation (23) the problem is fully solved. This is in stark contrast to the usual method of solving this problem where a specific model must be chosen in order to complete the analysis. If we choose some functional forms of $a(r,t)$ and $w(r,t)$ (which appear in both equations (23) and (24)), we are then restricting the solution to a particular model. There are a number of specific models in the literature which address this aspect of the problem. The models are not unanimous, and in practice, the popular models are those that result in closed-form solutions for the prices of basic bond-related derivatives.

Application of the method of multiple scales to the solution of partial differential equations (especially nonlinear ones) is a subtle matter (Nayfeh, 1994). In order to demonstrate the technique, we consider two popular single-factor interest rate models which are fundamentally different.

4.2 Ho & Lee Model (1986)

We begin by analysing the model of Ho & Lee (1986). The non-dimensional version of their model, equation (9), is:

$$dr(t) = \eta(t)dt + \epsilon^{\frac{1}{3}}\beta^{\frac{1}{2}}dX,$$

where β is a positive constant, and $\eta(t)$ is an, as yet unknown, non-dimensional time-dependent parameter. This equation represents the evolution of the instantaneous spot-rate in the risk-neutral environment. The function $\eta(t)$ is fitted to the initial yield curve, thereby ensuring that the calculated values of the discount functions (zero-coupon bond prices) for all maturities correspond to the initial market values.

It is important to note that if the Ho & Lee (1986) model has been developed within the risk-neutral environment, the function $\eta(t)$ represents the drift of the process for $r(t)$ in the risk-neutral environment. Therefore, equation (23) yields:

$$V_0(\xi, r_0, t) = -A(r_0, t_1, t_0)\frac{\beta}{2\eta(t_1, t_0)}f_1(\xi, t_1, t_0) + B(r_0, t_1, t_0), \tag{25}$$

where $f_1(\xi, t_1, t_0)$ is given by:

$$f_1(\xi, t_1, t_0) = e^{-\frac{2}{\beta}\eta(t_1,t_0)\xi} - 1.$$

Equation (24) is then simply:

$$\begin{aligned}\frac{\partial^2 V_1}{\partial \xi^2} + \frac{2}{\beta}\eta(t_1, t_0)\frac{\partial V_1}{\partial \xi} = -\frac{2}{\beta}\Bigg\{&-\frac{\beta}{2\eta(t_1, t_0)}f_1(\xi, t_1, t_0)\left(\frac{\partial A}{\partial t_1} - \frac{1}{\eta(t_1, t_0)}\left(\frac{\partial \eta}{\partial t_1}\right)A\right)\\ &+\frac{A}{2\eta(t_1, t_0)}\left(\frac{\partial \eta}{\partial t_1}\right)\xi e^{-\frac{2}{\beta}\eta(t_1,t_0)\xi} + \frac{\partial B}{\partial t_1}\Bigg\}. \end{aligned}\tag{26}$$

We do not need to solve (26). All we require is that the solution of (26) should be bounded and $\epsilon^{\frac{1}{3}}V_1$ should not be bigger than V_0 as $\xi \to \infty$. This solvability condition ensures that $V(r, t; \epsilon)$ is uniform and, ideally, it should determine the arbitrary functions $A(r_0, t_1, t_0)$ and $B(r_0, t_1, t_0)$.

We observe that the particular solution of (26), V_{1p}, is:

$$\begin{aligned}V_{1p}(\xi, r_0, t_1, t_0) = -\frac{\xi}{\eta(t_1, t_0)}\Bigg\{&\frac{\beta}{2\eta(t_1, t_0)}\left(\frac{\partial A}{\partial t_1} - \frac{1}{\eta(t_1, t_0)}\left(\frac{\partial \eta}{\partial t_1}\right)A\right)\left(e^{-\frac{2}{\beta}\eta(t_1,t_0)\xi} + 1\right)\\ &\frac{\beta}{2\eta(t_1, t_0)}\left(\frac{1}{\eta(t_1, t_0)} + \xi\right)e^{-\frac{2}{\beta}\eta(t_1,t_0)\xi}\left(\frac{\partial \eta}{\partial t_1}\right)A + \frac{\partial B}{\partial t_1}\Bigg\}.\end{aligned}\tag{27}$$

It can be shown, without too much trouble, that as $\xi \to \infty$:

$$\frac{V_{1p}}{V_0} \sim -\left\{\frac{\xi}{\eta}\frac{\left(\frac{\partial A}{\partial t_1} - \frac{1}{\eta}\left(\frac{\partial \eta}{\partial t_1}\right)A\right) + 2\eta\frac{\partial B}{\partial t_1}}{A\beta + 2\eta B}\right\}, \tag{28}$$

for some continuous functions $A(r_0, t_1, t_0)$ and $B(r_0, t_1, t_0)$. Hence, it follows from (27) that as $\xi \to \infty$, $\epsilon^{\frac{1}{3}}V_{1p}$ dominates V_0 unless:

$$\frac{\frac{\partial A}{\partial t_1}}{A(r_0, t_1, t_0)} - \frac{\frac{\partial \eta}{\partial t_1}}{\eta(t_1, t_0)} = 0, \tag{29}$$

$$\frac{\partial B}{\partial t_1} = 0. \tag{30}$$

Equation (29) solves to give:

$$A(r_0(t_1,t_0),t_1,t_0) = F_1(r_0(t_1,t_0),t_0)\eta(t_1,t_0), \tag{31}$$

for some arbitrary function $F_1(r_0(t_1,t_0),t_0)$. Equation (30) implies that B is a function of $r_0(t_0)$ and t_0 only:

$$B(r_0(t_1,t_0),t_1,t_0) = F_2(r_0(t_0),t_0). \tag{32}$$

Equations (31), (32), and (25) then yield:

$$V_0(\xi,r_0,t_1,t_0) = -F_1(r_0(t_0),t_0)\frac{\beta}{2}f_1\left(\xi,t_1,t_0\right) + F_2(r_0(t_0),t_0),$$

which, in original variables, is given by:

$$V_0(r(t),t) = -F_1(r(t),t)\frac{\beta}{2}\left(e^{-\frac{2}{\beta}\eta(t)(r_M-r(t))/\epsilon} - 1\right) + F_2(r(t),t). \tag{33}$$

The arbitrary functions $F_1(r(t),t)$ and $F_2(r(t),t)$ are then found by applying the boundary conditions. When $r = r_M$, $V_0(r,t) = V_m(r_M,t)$. Hence equation (33) gives:

$$V_m(r_M,t) = F_2(r_M,t),$$

which does not tell us, explicitly, what $F_2(r,t)$ is. However, if $V_m(r,t)$ and $F_2(r,t)$ are both continuous functions and have a one-to-one relationship for all values of r and t, we may infer that, in simple terms:

$$V_m(r,t) = F_2(r,t).$$

Thus, equation (33) becomes:

$$V_0(r(t),t) = V_m(r(t),t) - F_1(r(t),t)\frac{\beta}{2}\left\{e^{-\frac{2}{\beta}\eta(t)(r_M-r(t))/\epsilon} - 1\right\}. \tag{34}$$

On applying the boundary condition that $V_0(r,t) = V_M(r,t)$ at $r = r_m$, we get:

$$V_M(r_m,t) = V_m(r_m,t) - F_1(r_m,t)\frac{\beta}{2}\left\{e^{-\frac{2}{\beta}\eta(t)(r_M-r_m)/\epsilon} - 1\right\},$$

which implies that:

$$F_1(r_m,t) = -\frac{2\left(V_M(r_m,t) - V_m(r_m,t)\right)}{\beta\left(e^{-\frac{2}{\beta}\eta(t)(r_M-r_m)/\epsilon} - 1\right)}. \tag{35}$$

Consistent with condition (28), we then require $F_1(r(t),t)$ to be continuous at all times t and for all values of r. It is plausible then to insist that $F_1(r(t),t) = F_1(r_m,t)$ from equation (35). Hence equation (31) yields:

$$V_0(r,t) = V_m(r,t) + \frac{\left(V_M(r_m,t) - V_m(r_m,t)\right)\left(e^{-\frac{2}{\beta}\eta(t)(r_M-r(t))/\epsilon} - 1\right)}{\left(e^{-\frac{2}{\beta}\eta(t)(r_M-r_m)/\epsilon} - 1\right)}. \tag{36}$$

The application of the final condition $V_0(r,1)=1$ implies, as anticipated, that from (36) we have:

$$V_m(r,1) = V_M(r,1) = 1, \text{ for all } r_m \le r \le r_M.$$

Therefore, the non-dimensional solution $\bar{V}(\bar{r},\bar{t};\epsilon)$ is given by:

$$\bar{V}(\bar{r},\bar{t};\epsilon) = \bar{V}_m(\bar{r},\bar{t}) + \frac{\left(\bar{V}_M(\bar{r}_m,\bar{t}) - \bar{V}_m(\bar{r}_m,\bar{t})\right)\left(e^{-\frac{2}{\beta}\bar{\eta}(\bar{t})(\bar{r}_M-\bar{r}(\bar{t}))/\epsilon} - 1\right)}{\left(e^{-\frac{2}{\beta}\bar{\eta}(\bar{t})(\bar{r}_M-\bar{r}_m)/\epsilon} - 1\right)} + \cdots, \tag{37}$$

where we have re-introduced the bars here as a reminder that the equation is non-dimensional (i.e. all the "barred" variables are non-dimensional), and for the purpose of making the next step clearer.

In practice, the dimensional function $\eta(t)$ is obtained by calibrating the dimensional solution $V(r,t)$ with the market prices (which are also dimensional). Therefore it is important to re-dimensionalise (37) for it to be practically helpful. The dimensional version of (37) is given by:

$$V(r,t) = V_{min}(r,t) + \frac{\left(V_{max}(r_{min},t) - V_{min}(r_{min},t)\right)\left(e^{-\frac{2}{\beta}\eta(t)(r_{max}-r(t))} - 1\right)}{\left(e^{-\frac{2}{\beta}\eta(t)(r_{max}-r_{min})} - 1\right)} + \cdots, \tag{38}$$

where, we recall, that $V(r,t) = V_T\bar{V}(\bar{r},\bar{t})$ (V_T is the known maturity value of the zero-coupon bond), $r = R\bar{r}$ (R is the typical risk-free interest rate) and $t = T\bar{t}$ (T is the time to maturity, in years, of the bond). It can be verified that (38) is dimensionally correct, consistent with the dimensions of β, $[\beta] = 1/\left(R^2T^5\right)$ and the term $\eta(t)$, whose dimensions are $[\eta(t)] = T/R$.

4.3 Vasicek (1977) Model

As a second example, we consider the model of Vasicek (1977). Under the conditions of interest, the non-dimensional version of the Vasicek (1977) model is:

$$dr(t) = \alpha\left(\gamma - r(t)\right)dt + \epsilon^{\frac{1}{3}}\beta^{\frac{1}{2}}dX,$$

where α, γ, and β are all positive constants. With this characterisation of the process, equation (23) gives:

$$V_0(\xi,r_0,t_1,t_0) = -A(r_0,t_1,t_0)\frac{\beta}{2\alpha(\gamma - r_0(t_0))}f_1(\xi,r_0,t_0) + B(r_0,t_1,t_0), \tag{39}$$

where $f_1(\xi,r_0,t_0)$ is given by:

$$f_1(\xi,r_0,t_0) = e^{-\frac{2\alpha}{\beta}(\gamma - r_0(t_0))\xi} - 1.$$

The functions $A(r_0,t_1,t_0)$ and $B(r_0,t_1,t_0)$ are obtained by requiring that (39) should be asymptotically uniform with respect to the solution of:

$$\frac{\partial^2 V_1}{\partial\xi^2} + \frac{2\alpha}{\beta}(\gamma - r_0(t_0))\frac{\partial V_1}{\partial\xi} = -\frac{2}{\beta}\left\{-\left(\frac{\partial A}{\partial t_1}\right)\frac{\beta}{2\alpha(\gamma - r_0(t_0))}f_1(\xi,r_0,t_0) + \frac{\partial B}{\partial t_1}\right\}, \tag{40}$$

as $\xi \to \infty$.

(It is particularly important to notice that, in this case, $r = r(t_0)$ only. This is a consequence of equation (10), $dr/dt = u(r,t)$. That is to say:

$$\frac{1}{\epsilon}\frac{\partial r_0}{\partial t_1} + \frac{\partial r_0}{\partial t_0} = u(r_0, t_1, t_0),$$

which, to lowest order of ϵ, implies that $\partial r_0/\partial t_1 = 0$.)

Now, the particular solution of (40) is:

$$V_{1p} = -\frac{\xi}{\alpha(\gamma - r_0(t_0))}\left\{\frac{\beta}{2\alpha(\gamma - r_0(t_0))}\left(\frac{\partial A}{\partial t_1}\right) f_1(\xi, r_0, t_0) + \frac{\partial B}{\partial t_1}\right\}.$$

Thus, (39) is uniform if both A and B are functions of r_0 and t_0 only, i.e.:

$$\begin{aligned}\frac{\partial A}{\partial t_1} &= 0,\\ \frac{\partial B}{\partial t_1} &= 0.\end{aligned}$$

Hence, in original variables, equation (39) is given by:

$$V_0(r,t) = -A(r,t)\frac{\beta}{2\alpha(\gamma - r(t))}\left(e^{-\frac{2\alpha}{\beta}(\gamma - r(t))(r_M - r(t))/\epsilon} - 1\right) + B(r,t). \tag{41}$$

The application of the boundary conditions that $V_0 = V_M(r,t)$ at $r = r_m$, and $V_0 = V_m(r,t)$ when $r = r_M$ implies that $A(r_m,t)$ and $B(r_M,t)$ are respectively given by:

$$\begin{aligned}A(r_m,t) &= -\frac{2\alpha(\gamma - r_m)\left(V_M(r_m,t) - B(r_m,t)\right)}{\beta\left(e^{-\frac{2\alpha}{\beta}(\gamma - r_m)(r_M - r_m)/\epsilon} - 1\right)},\\ B(r_M,t) &= V_m(r_M,t).\end{aligned}$$

These suggest that the non-dimensional solution $\bar{V}_0(\bar{r},\bar{t})$ is given by:

$$\bar{V}_0(\bar{r},\bar{t}) = \bar{V}_m(\bar{r},\bar{t}) + \left\{\bar{V}_M(\bar{r}_m,\bar{t}) - \bar{V}_m(\bar{r}_m,\bar{t})\right\}\frac{(\bar{\gamma} - \bar{r}_m)\left(e^{-\frac{2\bar{\alpha}}{\beta}(\bar{\gamma} - \bar{r}(\bar{t}))(\bar{r}_M - \bar{r}(\bar{t}))/\epsilon} - 1\right)}{(\bar{\gamma} - \bar{r}(\bar{t}))\left(e^{-\frac{2\bar{\alpha}}{\beta}(\bar{\gamma} - \bar{r}_m)(\bar{r}_M - \bar{r}_m)/\epsilon} - 1\right)}, \tag{42}$$

where $A(\bar{r}_m,\bar{t}) = A(\bar{r},\bar{t})$ so that $A(\bar{r},\bar{t})$ is continuous for all values of $\bar{r}$, $\bar{t}$ and, for simplicity, $B(\bar{r},\bar{t}) = \bar{V}_m(\bar{r},\bar{t})$.

Therefore, the dimensional solution $V(r,t)$ is given by:

$$V(r,t) = V_{min}(r,t) + \{V_{max}(r_{min},t) - V_{min}(r_{min},t)\}\frac{(\gamma - r_{min})\left(e^{-\frac{2\alpha}{\beta}(\gamma - r(t))(r_{max} - r(t))} - 1\right)}{(\gamma - r(t))\left(e^{-\frac{2\alpha}{\beta}(\gamma - r_{min})(r_{max} - r_{min})} - 1\right)} + \cdots, \tag{43}$$

where the dimensions of the term α/β are $[\alpha/\beta] = 1/\left(R^2T^5\right)$, while those of $(\gamma - r(t))$ are $[(\gamma - r(t))] = T/R$.

5 Short-Term-to-Maturity Bond

A completely different problem, with separate dynamics, presents itself when we investigate a STTM solution of equation (6). If we assume that the typical time to maturity $T \sim 2$ years, and the typical risk-free interest rate $R \sim 12\%$, then the dimensionless parameter $RT \sim 0.24$. Thus, we have $(RT)^3 \sim 0.01$ which is "small". We therefore write $(RT)^3 = \epsilon$, for some small positive parameter ϵ. Under these circumstances, we obtain the following non-dimensional equation:

$$\epsilon^{\frac{2}{3}} \frac{\partial V}{\partial t} + \frac{w^2}{2} \frac{\partial^2 V}{\partial r^2} - \epsilon a \frac{\partial V}{\partial r} - \epsilon r V = 0, \tag{44}$$

from equation (6).

Equation (44) is a regular perturbation problem because the highest order derivative term is not of $O(\epsilon)$. To leading order, this equation can be solved by seeking a standard asymptotic expansion for V. However, a solution to the full problem does not simply involve solving equation (44) to leading order. We require the solution to equation (44) in conjunction with equation (7). On examination, equation (7):

$$dr(t) = u(r,t)dt + \frac{1}{\epsilon^{1/3}} w(r,t)dX, \tag{45}$$

suggests that r should be re-scaled as:

$$r = \frac{\hat{r}}{\epsilon^{1/3}}.$$

This transforms equations (45) and (44), respectively, to:

$$\epsilon^{\frac{1}{3}} u(\hat{r},t)dt + w(\hat{r},t)dX = dr_1, \tag{46}$$

$$\epsilon^{\frac{2}{3}} \frac{\partial V}{\partial t} + \epsilon^{\frac{2}{3}} \frac{w^2}{2} \frac{\partial^2 V}{\partial \hat{r}^2} - \epsilon^{\frac{4}{3}} a \frac{\partial V}{\hat{r}} - \hat{r} V = 0. \tag{47}$$

In order to have a non-trivial solution to leading order, equation (47) suggests a re-scaling of t, and hence of X, as:

$$t = \epsilon^{\frac{2}{3}} \hat{t},$$

$$X = \epsilon^{\frac{2}{3}} \hat{X}.$$

Thus, equation (46) reduces to:

$$d\hat{r}(\hat{t}) = \epsilon^{\frac{1}{3}} \epsilon^{\frac{2}{3}} u(\hat{r},\hat{t}) d\hat{t} + \epsilon^{\frac{2}{3}} w(\hat{r},\hat{t}) d\hat{X}. \tag{48}$$

Equation (48) suggests further that (consistent with the initial scalings of u and w) we should re-scale u and w as:

$$u = \epsilon^{-\frac{2}{3}} \hat{u},$$

$$w = \epsilon^{-\frac{2}{3}} \hat{w},$$

which leads, finally, to:

$$d\hat{r}(\hat{t}) = \epsilon^{\frac{1}{3}}\hat{u}(\hat{r},\hat{t})d\hat{t} + \hat{w}(\hat{r},\hat{t})d\hat{X}. \tag{49}$$

(It is interesting to note that in this case, to leading order, the variation in the short-term interest rate is driven solely by the Brownian motion, in direct contrast to equation (10) in Section 4.) Equation (47) then becomes:

$$\frac{\partial V}{\partial \hat{t}} + \frac{\hat{w}^2}{2}\frac{\partial^2 V}{\partial \hat{r}^2} - \epsilon^{\frac{4}{3}}a\frac{\partial V}{\partial \hat{r}} - \hat{r}V = 0. \tag{50}$$

Equation (50) is still a regular perturbation problem because the highest order derivative term is not of $O(\epsilon)$. To leading order, we have:

$$\frac{\partial V_0}{\partial \hat{t}} + \frac{\hat{w}^2}{2}\frac{\partial^2 V_0}{\partial \hat{r}^2} - \hat{r}V_0 = 0, \tag{51}$$

where V has been expanded as:

$$V = V_0 + \epsilon^{\frac{4}{3}}V_1 + \epsilon^{\frac{8}{3}}V_2 + \cdots.$$

The correction terms, V_i $(i = 1, 2, \cdots)$, are obtained by iteratively solving the equations:

$$\frac{\partial V_i}{\partial \hat{t}} + \frac{\hat{w}^2}{2}\frac{\partial^2 V_i}{\partial \hat{r}^2} = a\frac{\partial V_{i-1}}{\partial \hat{r}},$$

where the right hand side is always known *a priori*. We are initially interested in obtaining the leading order term V_0.

It is unsurprising that equation (51) is identical to the Black (1976) equation governing derivatives written on futures. To leading order, equation (49) implies that $\hat{r}$ is a martingale under either the risk-neutral or real-world measures. Consequently, the differential equation reduces automatically to the form of equation (51).

We will solve equation (51) subject to (we recall) the following boundary conditions: maturity $\hat{t} = \tau$ (where, now, $\tau = \epsilon^{-2/3}$) and $V_0(\hat{r},\hat{t}) = 1$; the bond is anticipated to become valueless should the short-term interest rate escalate to infinity, i.e. $V(\hat{r},\hat{t}) \to 0$ as $\hat{r} \to \infty$ (that is to say, from a practical point of view, it is ideally sensible to expect that people, in their right mind, are extremely unlikely to buy the bond when they could be making more money by simply packing their money in a bank account); another boundary condition is that when $\hat{r} \to 0$, $V(\hat{r},\hat{t}) = \psi(\hat{t})$, for some known function $\psi(\hat{t})$.

5.1 A Constant Volatility Problem: $\hat{w}^2 = \beta^2$, a constant

Since no one is likely to make any progress until they have first identified and worked out the simplest possible cases, it is instructive, therefore, to proceed by studying the problem when $\hat{w}(\hat{r},\hat{t})$

is simply a constant, β, say. (Although, from our perspective, this is simply a paradigm problem, it may be worth mentioning that in fact the standard Vasicek (1977) and Ho & Lee (1986) models fall under this category.)

A conventional approach to solving equation (51) in the literature would begin by assuming that the volatility, $\hat{w}(\hat{r}, \hat{t})$, is known precisely, either from compelling economic arguments or tractability-based arguments, or even from emperical foundations. Another common approach is to parameterise $\hat{w}(\hat{r}, \hat{t})$ and then assume that $\hat{w}^2(\hat{r}, \hat{t})$ is linear in $\hat{r}$. Thus, a simple solution of the form $\exp\big(A(\hat{t}, \tau) + \hat{r}B(\hat{t}, \tau)\big)$, for some functions $A(\hat{t}, \tau)$ and $B(\hat{t}, \tau)$, is admissible in equation (51). Then the solution of the problem boils down to determining appropriate functional forms of $A(\hat{t}, \tau)$ and $B(\hat{t}, \tau)$ in terms of relevant parameters, by fitting market yield curves.

There is probably nothing wrong with any of these approaches (for, it seems that – in practice – the volatility is only known to the extent that it may be estimated from historical data, or by fitting models' prices to the prices currently trading in the market); but there seems to be no unique procedure of doing this in the literature. Moreover, volatility – having been estimated by any of these methods – seems to never stay the same; it appears to change in a completely unpredictable fashion (and attempts to model it as a stochastic process also lead to hedging difficulties, since volatility is not a traded asset). Evidently, volatility is an interesting – and still a subtle – phenomenon.

In this paper, we propose that the volatility, $\hat{w}(\hat{r}, \hat{t})$, be obtained as part of solution to the problem. In particular, when $\hat{w}(\hat{r}, \hat{t})$ is a constant parameter, this is simply an eigenvalue problem. The eigenvalues will dictate which values of the volatility are admissible for this problem (because it is possible that not every arbitrarily prescribed value of the volatility is suitable; but it is very unlikely for anyone to conclusively detect this through just fitting the models' output to the market yield curves, or by simply looking at historical data (though this does not mean that this practice is not useful or profitable – this may be a very subtle matter)).

Thus, our attitude is very simple: that equation (51) does not necessarily have a sensible solution for all arbitrarily prescribed values of $\hat{w} = \beta$ (a constant parameter). In other words, β should be found as an eigenvalue to this problem. In order to achieve this, we reduce the problem to solving second-order ordinary differential equations in $\hat{r}$, of which there is a wealth of knowledge in the literature. We do so by looking for solutions of the form:

$$V_0(\hat{r}, \hat{t}) = F(\hat{r}) + f(\hat{r}, \hat{t}). \tag{52}$$

We then intuitively insist that $f(\hat{r}, \hat{t})$ must be a solution of:

$$\frac{\partial f}{\partial \hat{t}} + \frac{\beta^2}{2}\frac{\partial^2 f}{\partial \hat{r}^2} - \hat{r}f = 0.$$

This implies that $f(\hat{r},\hat{t})$ may be written as:

$$f(\hat{r},\hat{t}) = b\exp\left\{\frac{\beta^2}{6}(\tau-\hat{t})^3 - \hat{r}(\tau-\hat{t})\right\}, \tag{53}$$

for, as yet to be found, constant $b > 0$.

The function $F(\hat{r})$ and the associated eigenvalues, β, must be obtained by solving the ordinary differential equation:

$$\frac{\beta^2}{2}\frac{d^2F}{d\hat{r}^2} - \hat{r}F = 0, \tag{54}$$

subject to these boundary conditions: At maturity ($\hat{t} = \tau$), $V_0(\hat{r},\hat{t}) = 1$. When $\hat{r} = 0$, we require that $F(\hat{r}) = 0$ so that $V_0(0,\hat{t}) = \psi(\hat{t}) = b\exp\left\{\beta^2(\tau-\hat{t})^3\right\}/6$. As $\hat{r} \to \infty$, the bond becomes valueless, i.e. $V_0(\hat{r},\hat{t}) \to 0$, and hence $F(\hat{r})$ must tend to zero, $F(\hat{r}) \to 0$. (It may be important to note, at this stage, that by prescribing a boundary condition at $\hat{r} = 0$ we do not, in any way, preclude negative values of $\hat{r}$. Afterall, in some real world circumstances, such as that in Switzerland in the 1960s, interest rates do become negative (Wilmott, 1998).)

With these boundary conditions, we will restrict ourselves – for simplicity and briefty – to only solving equation (54) for $F(\hat{r})$ when β is vanishingly small or very large. In this case, the problem becomes a standard asymptotic analysis problem. It is also important to observe that whether $\beta \to 0$ or $\beta \to \infty$, the asymptotic analysis for this problem is qualitatively the same. Hence, we proceed by considering only the case when $\beta \to 0$ in details, and we will simply state the result for the case when $\beta \to \infty$.

In the case that $\beta \to 0$, equation (54) is a standard singular perturbation problem where the smallness parameter β is multiplying the highest derivative term. The problem may then be solved by standard WKB method (see for example, Nayfeh, 1994 and Hinch, 1991) to obtain an "outer" solution, $F^O(\hat{r};\beta)$:

$$F^O(\hat{r};\beta) = (2\hat{r})^{-1/4}\left\{A\exp\left(\frac{2}{3}\sqrt{2}\frac{1}{\beta}\hat{r}^{3/2}\right) + B\exp\left(-\frac{2}{3}\sqrt{2}\frac{1}{\beta}\hat{r}^{3/2}\right)\right\}, \tag{55}$$

where A and B are arbitrary constants (and they may possibly depend parametrically on τ). The outer solution is clearly not valid as $\hat{r} \to 0$; therefore, it will have to be matched with an "inner" solution, $F^I(\hat{r};\beta)$:

$$F^I(\xi) = CAi(\xi) + DBi(\xi), \tag{56}$$

(which is valid in the region near and at $\hat{r} = 0$) where $\xi = 2^{1/3}\hat{r}\beta^{-2/3}$. The arbitrary constants C and D, which are related to A and B, should be determined by matching equations (55) and (56). Finally, the application of boundary conditions should determine suitable values of β. The functions $Ai(\xi)$ and $Bi(\xi)$ are the Airy functions of the first- and the second-kind, respectively; they are usually defined as:

$$Ai(\xi) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{1}{3}\eta^3 + \xi\eta\right)d\eta,$$

$$Bi(\xi) = \frac{1}{\pi}\int_0^\infty \left\{\exp\left(-\frac{1}{3}\eta^3+\xi\eta\right)+\sin\left(\frac{1}{3}\eta^3+\xi\eta\right)\right\}d\eta.$$

The matching process for equations (55) and (56), explained above, is easier said than done; in reality, it is cumbersome and clumsy. As an alternative, a solution which is valid globally (i.e. both in the neighbourhood of, and far away from, the point $\hat{r}=0$), may be obtained through the use of standard Langer transformation (see for example, Nayfeh, 1994). In summary, Langer transformation changes both the dependent and independent variables as:

$$x = \phi(r),$$
$$y(x) = \psi(\hat{r})F(\hat{r}),$$

where ϕ and ψ are chosen to ensure that the dominant part of the transformed equation is simply the Airy equation:

$$\frac{d^2y}{dx^2} - xy = 0, \tag{57}$$

whose solutions have qualitatively the same behaviour as those of equation (54). This ensures that $d\phi/d\hat{r}$, which is given by:

$$\frac{4}{\beta^2}\hat{r} = \left(\frac{d\phi}{d\hat{r}}\right)^2 x, \tag{58}$$

is regular and has no zeros in the interval of interest and, moreover, ϕ satisfies the equation:

$$\frac{d^2\phi}{d\hat{r}^2} - \frac{2}{\psi}\frac{d\phi}{d\hat{r}}\frac{d\psi}{d\hat{r}} = 0. \tag{59}$$

It is consistent with the analysis and the form of the dominant part of the transformed equation (57) that the term:

$$-\frac{1}{\psi}\left(\frac{d\hat{r}}{d\phi}\right)^2\left\{\frac{d^2\psi}{d\hat{r}^2}+\frac{2}{\psi}\left(\frac{d\psi}{d\hat{r}}\right)^2\right\},$$

is negligible. The function ψ is related to ϕ, from equation (59), by:

$$\psi = \left(\frac{d\phi}{d\hat{r}}\right)^{1/2}.$$

Since $x=\phi$, equation (58) implies that:

$$\phi^{3/2} = \pm\frac{2}{\beta}\hat{r}^{3/2}.$$

Furthermore, we have:

$$\psi = \left(\frac{d\phi}{d\hat{r}}\right)^{1/2} = \pm\left(\frac{2}{\beta}\right)^{1/2}\left(\frac{\hat{r}}{x}\right)^{1/4}.$$

The general solution for equation (57) may be written as:

$$y(x) = \bar{E}_1 Ai(x) + \bar{E}_2 Bi(x).$$

Thus, the required global solution is given by:

$$F(\hat{r}) = \left(\frac{x}{\hat{r}}\right)^{1/4}\left(E_1 Ai(x) + E_2 Bi(x)\right), \tag{60}$$

where $\pm(2/\beta)^{1/2}$ has been absorbed into the constants of integration E_1 and E_2.

The next thing is, obviously, to apply the boundary conditions. On observing that as $\hat{r} \to \infty$ (with $\beta \to 0$), $x \to \infty$, and applying the boundedness condition that $F(\hat{r}) \to 0$ as $\hat{r} \to \infty$, we then conclude that E_2 must be zero. This is simply derived from the fact that as $x \to \infty$ (see for example, Albowitz & Clarkson, 1991), $Ai(x)$ and $Bi(x)$ have asymptotic behaviours:

$$\begin{aligned} Ai(x) &\sim \frac{1}{2\pi^{1/2}x^{1/4}} \exp\left\{-\frac{2}{3}x^{3/2}\right\}, \\ Bi(x) &\sim \frac{1}{\pi^{1/2}x^{1/4}} \exp\left\{\frac{2}{3}x^{3/2}\right\}. \end{aligned}$$

Furthermore, by imposing the boundary condition $F(0) = 0$, we must have:

$$Ai(x(0)) = 0, \tag{61}$$

where, we know, that x is a function of β. Thus, the roots of equation (61) must yield the appropriate eigenvalues. Moreover, on recalling that:

$$d\phi = dx = -\frac{2}{3}\frac{\hat{r}}{\beta^{5/3}}d\beta + \frac{1}{\beta^{2/3}}d\hat{r},$$

it can be easily shown that as $\hat{r} \to 0$ (with $\beta \to 0$), then $x \to -\infty$. As a result, we must have (see for example, Albowitz & Clarkson, 1991 and Nayfeh, 1994):

$$Ai(x(0)) \sim \frac{1}{\sqrt{\pi}(x(0))^{1/4}} \sin\left\{\frac{2}{3}(-x(0))^{3/2} + \frac{\pi}{4}\right\}.$$

Hence, we get:

$$\sin\left\{\frac{2}{3}\left(\frac{2}{3}\frac{1}{\beta^{5/3}}\right)^{3/2} + \frac{\pi}{4}\right\} = 0,$$

which gives:

$$\frac{2}{3}\left(\frac{2}{3}\frac{1}{\beta^{5/3}}\right)^{3/2} + \frac{\pi}{4} = n\pi,$$

where $n = 0, 1, 2, \cdots$. The eigenvalues, β, are therefore given by:

$$\beta = \left(\frac{2}{3}\left\{\frac{2}{3}\pi\left(n - \frac{1}{4}\right)\right\}^{-2/3}\right)^{3/5}. \tag{62}$$

Therefore, the required approximate solution for the price of the zero-coupon bond, $V_0(\hat{r}, \hat{t})$, is given in closed-form as:

$$V_0(\hat{r}, \hat{t}) = E_1\left(\frac{2}{\beta}\right)^{2/3} Ai\left(\left(\frac{2}{\beta}\right)^{2/3}\hat{r}\right) + b\exp\left\{\frac{\beta^2}{6}(\tau - t)^3 - \hat{r}(\tau - \hat{t})\right\}. \tag{63}$$

where, clearly, the boundary condition at $\hat{t} = \tau$ demands that $b = 1$, and E_1 may be suitably defined by some generalised functions: $H_1(\hat{t} - \tau)$ and $H_2(\hat{t} - \tau)$, where $H_1(\hat{t} - \tau) = 1$ if $\hat{t} < \tau$ and zero otherwise; $H_2(\hat{t} - \tau) = 1$ if $\hat{t} > \tau$ and zero otherwise. Hence, for all $\hat{t}$, $V_0(\hat{r}, \hat{t})$ may be written as:

$$V_0(\hat{r}, \hat{t}) = \left(H_1(\hat{t} - \tau) + H_2(\hat{t} - \tau)\right)\left(\frac{2}{\beta}\right)^{2/3} Ai\left(\left(\frac{2}{\beta}\right)^{2/3}\hat{r}\right) + \exp\left\{\frac{\beta^2}{6}(\tau - \hat{t})^3 - \hat{r}(\tau - \hat{t})\right\}. \tag{64}$$

Now, we will state the result when β is allowed to grow very large, $\beta \to \infty$. In this case, equation (54) looks like a regular perturbation problem at first. However, on expanding F asymptotically in powers of β and solving the leading order problem with appropriate boundary conditions, we get a trivial solution. This is clearly not correct; and the reason is simply that if we define a new variable χ by $\chi = \hat{r}/\beta$ we get:

$$\frac{1}{2\beta^2}\frac{d^2F}{d\chi^2} - \chi F = 0,$$

where $1/\beta^2 \to 0$. The analysis then becomes identical to the one for the previous case. We obtain the eigenvalues:

$$\beta = \left(\frac{2}{3}\left\{\frac{2}{3}\pi\left(n - \frac{1}{4}\right)\right\}^{-2/3}\right)^{-3/5}, \tag{65}$$

for $n = 0, 1, 2, \cdots$.

It is possible that the solution we have constructed here has never – and perhaps will never be – observed in practice (besides, by our own admission, we have mentioned earlier that in the real world volatility seems to be strictly a function of time). But, at the same time, this result, simple as it is, is suggests that, for single-factor models, the idea of one "correct" fair price for the zero-coupon bond, when the time to maturity is "short", may be an illusion. For, even in this very simple case of constant volatility (once we insist that the volatility cannot be arbitrarily prescribed, but that it must be found as part of solution to the problem), we obtain a narrow region of a family of infinitely many bounded solutions that all intersect at maturity, when $\hat{t} = \tau$. (From a practical point of view, this result loosely implies that, in general, as long as different prices for one instrument are not too different, at one given time, the market – as is evide! ntly true in reality – will not make too much of a fuss and will tolerate them.) However, our main interest is to investigate the problem when the volatility is strictly a function of time, like it appears to be in the real world.

5.2 A Non-Constant Volatility, $\hat{w}(\hat{r}, \hat{t}) > 0$, Problem

In this section, we analyse a supposedly more realistic problem where the volatility is neither a constant nor a parameter. Following the style approach adopted in the preceding section,we look for a solution, for equation (51), of the form:

$$V_0(\hat{r}, \hat{t}) = F(\hat{r}, \hat{t}) + P(\hat{r}, \hat{t}) \exp\left\{\frac{\hat{w}^2}{6}(\tau - \hat{t})^3 - \hat{r}(\tau - \hat{t})\right\}, \tag{66}$$

where $F(\hat{r}, \hat{t})$, $P(\hat{r}, \hat{t})$ and $\hat{w}(\hat{r}, \hat{t})$ are to be determined.

For this solution to be consistent with the prescribed boundary conditions, it is mandatory that $P(\hat{r}, \hat{t})$ be a constant, P_0 (say). We then obtain $\hat{w}(\hat{r}, \hat{t})$ by insisting that:

$$P_0 \exp\left\{\frac{\hat{w}^2}{6}(\tau - \hat{t})^3 - \hat{r}(\tau - \hat{t})\right\},$$

be a non-trivial and non-constant particular solution of equation (51). This fixes the form of $\hat{w}(\hat{r},\hat{t})$, which has to be obtained by solving the partial differential equation:

$$\frac{\hat{w}}{3}\frac{\partial\hat{w}}{\partial\hat{t}} + \frac{\hat{w}^2}{6}\left(\frac{\partial\hat{w}}{\partial\hat{r}}\right)^2 + \frac{\hat{w}^3}{6}\frac{\partial^2\hat{w}}{\partial\hat{r}^2} + \frac{\hat{w}^4}{18}\left(\frac{\partial\hat{w}}{\partial\hat{r}}\right)^2(\tau-\hat{t})^3 - \frac{\hat{w}^3}{3}\frac{\partial\hat{w}}{\partial\hat{r}}(\tau-\hat{t}) = 0. \tag{67}$$

It is easy to show or to verify from equation (67) that $\hat{w}(\hat{r},\hat{t})$ is neither a function of $\hat{r}$ only nor a function of $\hat{t}$ only. On assuming, then, that $\hat{w}(\hat{r},\hat{t})$ may be written as $\hat{w}(\hat{r},\hat{t}) = \mathcal{R}(\hat{r})\mathcal{T}(\hat{t})$, we thus have to solve the ordinary differential equation:

$$\frac{1}{3\mathcal{T}^5(\tau-\hat{t})^3}\frac{d\mathcal{T}}{d\hat{t}} + \frac{1}{6\mathcal{T}^2(\tau-\hat{t})^3}\left\{\left(\frac{d\mathcal{R}}{d\hat{r}}\right)^2 + R\frac{d^2\mathcal{R}}{d\hat{r}^2}\right\} + \frac{\mathcal{R}^2}{18}\left(\frac{d\mathcal{R}}{d\hat{r}}\right)^2 - \frac{\mathcal{R}}{3\mathcal{T}^2(\tau-\hat{t})^2}\frac{d\mathcal{R}}{d\hat{r}} = 0. \tag{68}$$

Equation (68) has a non-obvious and non-trivial solution, for some constants $a \neq 0$ and $b \geq 0$:

$$\mathcal{R}(\hat{r}) = (a(\hat{r}+b))^{1/2},$$

where $\mathcal{T}(\hat{t})$ has to be obtained by solving the first order ordinary differential equation:

$$24d\mathcal{T} - \left(-a^2\mathcal{T}^5(\tau-\hat{t})^3 + 12a\mathcal{T}^3(\tau-\hat{t})\right)dt = 0. \tag{69}$$

Equation (69) may then be solved to yield:

$$\mathcal{T}(\hat{t}) = A(\tau-\hat{t})^{-1},$$

where the constant A is obtained from the algebraic equation:

$$a^2A^4 - 12aA^2 + 24 = 0. \tag{70}$$

Equation (70) solves to give:

$$A = \pm\sqrt{\frac{6\pm 2\sqrt{3}}{a}},$$

which gives a real and positive volatility, $\hat{w}(\hat{r},\hat{t})$, given by:

$$\hat{w}_{\pm}(\hat{r},\hat{t}) = \sqrt{6\pm 2\sqrt{3}}\frac{(\hat{r}+b)^{1/2}}{\tau-\hat{t}}. \tag{71}$$

The constant parameter b is obtained along with the solution for $F(\hat{r},\hat{t})$ from the de-coupled partial differential equation:

$$\frac{\partial F_{\pm}}{\partial\hat{t}} + \frac{6\pm 2\sqrt{3}}{2}\frac{\hat{r}+b}{(\tau-\hat{t})^2}\frac{\partial^2 F_{\pm}}{\partial\hat{r}^2} - \hat{r}F_{\pm} = 0. \tag{72}$$

(It may be important, before we proceed, to comment here about the form of the volatility, $\hat{w}(\hat{r},\hat{t})$, in equation (71). In the conventional way of solving this problem in the literature, it is generally normal to parameterise $\hat{w}(\hat{r},\hat{t})$; and the parameters are carefully chosen so that the theoretical price of the zero-coupon bond exactly matches the price currently being reflected by the market. In particular, it is almost always assumed that $\hat{w}^2(\hat{r},\hat{t})$ is linear in $\hat{r}$; and it is interesting to note this form rearing its head here naturally from this simple mathematical analysis.)

5.2.1 Determination of b and $F(\hat{r},\hat{t})$

Equation (72) may be attacked by a brute force of numerical computations, and this style of approach is often popular. However, because the solution may not necessarily exist for every arbitrary value of b, it is preferable to present a closed-form solution and simultaneously determine which values of b are admissible.

It is not easy to develop a natural intuition for the general solution of equation (72). However, on noting that the differential equation:

$$\frac{6\pm 2\sqrt{3}}{2}\frac{\hat{r}}{\tau^2}\frac{\partial^2 F_{s\pm}}{\partial \hat{r}^2} - \hat{r}F_{s\pm} = 0,$$

has a bounded general solution, as $\hat{r}\to\infty$, given by:

$$F_{s\pm}(r) = B_{\pm}\exp\left(-\sqrt{\frac{2}{6\pm 2\sqrt{3}}}\tau\hat{r}\right); \quad B_{\pm}\neq 0,$$

we then expect that, if b is not too large, the general solution of equation (72) may be given by:

$$F_{\pm}(r,t) = B_{\pm}(\hat{r},\hat{t})\exp\big(-C_{\pm}(\hat{r},\hat{t})\big), \tag{73}$$

where $B_{\pm}(\hat{r},\hat{t})$ and $C_{\pm}(\hat{r},\hat{t})$ are to be determined. A substitution of equation (73) into (72) gives:

$$\frac{\partial}{\partial \hat{t}}(B_{\pm}-C_{\pm}) + \frac{6\pm 2\sqrt{3}}{2}\frac{(\hat{r}+b)}{(\tau-\hat{t})^2}\left\{\frac{\partial^2 B_{\pm}}{\partial \hat{r}^2} - 2\frac{\partial B_{\pm}}{\partial \hat{r}}\frac{\partial C_{\pm}}{\partial \hat{r}} - B_{\pm}\frac{\partial^2 C_{\pm}}{\partial \hat{r}^2} + B_{\pm}\left(\frac{\partial C_{\pm}}{\partial \hat{r}}\right)^2\right\} - \hat{r}B_{\pm} = 0,$$

which, upon assuming that $B_{\pm}(\hat{r},\hat{t}) = \mathcal{T}_{1\pm}(\hat{t})\mathcal{R}_{1\pm}(\hat{r})$ and $C_{\pm}(\hat{r},\hat{t}) = \mathcal{T}_{2\pm}(\hat{t})\mathcal{R}_{2\pm}(\hat{r})$, leads to the ordinary differential equation:

$$\left\{\frac{d}{d\hat{t}}\left(\frac{\mathcal{R}_{1\pm}}{\mathcal{R}_{2\pm}}\mathcal{T}_{1\pm} - \mathcal{T}_{2\pm}\right)\right\}(\tau-\hat{t})^2 - \left\{\frac{6\pm 2\sqrt{3}}{2}\frac{(\hat{r}+b)}{\mathcal{R}_{2\pm}}\frac{d^2\mathcal{R}_{1\pm}}{d\hat{r}^2} + \hat{r}\frac{\mathcal{R}_{1\pm}}{\mathcal{R}_{2\pm}}(\tau-\hat{t})^2\right\}\mathcal{T}_{1\pm} + (6\pm 2\sqrt{3})(\hat{r}+b)\left(\frac{1}{\mathcal{R}_{2\pm}}\frac{d\mathcal{R}_{1\pm}}{d\hat{r}}\frac{d\mathcal{R}_{2\pm}}{d\hat{r}}\mathcal{T}_{1\pm} + \frac{1}{2}\frac{\mathcal{R}_{1\pm}}{\mathcal{R}_{2\pm}}\left(\frac{d^2\mathcal{R}_{2\pm}}{d\hat{r}^2} - \left(\frac{d\mathcal{R}_{2\pm}}{d\hat{r}}\right)^2\mathcal{T}_{1\pm}\mathcal{T}_{2\pm}\right)\right)\times \mathcal{T}_{2\pm} = 0.$$

Then, a close inspection of this equation reveals that:

$$\frac{\mathcal{R}_{1\pm}}{\mathcal{R}_{2\pm}} = \frac{\kappa_{1\pm}}{\hat{r}}, \tag{74}$$

$$\mathcal{T}_{1\pm} = \kappa_{2\pm}, \tag{75}$$

$$\mathcal{T}_{2\pm} = \kappa_{3\pm}(\tau-\hat{t}), \tag{76}$$

for some arbitrary constants $\kappa_{1\pm}$, $\kappa_{2\pm}$, and $\kappa_{3\pm}$. That is to say:

$$\left\{-\kappa_{3\pm} - \kappa_{1\pm}\kappa_{2\pm} - \frac{6\pm 2\sqrt{3}}{2}\frac{(\hat{r}+b)}{\hat{r}}\kappa_{1\pm}\kappa_{2\pm}\kappa_{3\pm}^2\left(\frac{d\mathcal{R}_{2\pm}}{d\hat{r}}\right)^2\right\}(\tau-\hat{t})^2 - \left\{\frac{6\pm 2\sqrt{3}}{2}\frac{(\hat{r}+b)}{\mathcal{R}_{2\pm}}\frac{d^2\mathcal{R}_{1\pm}}{d\hat{r}^2}\right\} + \left\{(6\pm 2\sqrt{3})(\hat{r}+b)\left(\frac{\kappa_{2\pm}}{\mathcal{R}_{2\pm}}\frac{d\mathcal{R}_{1\pm}}{d\hat{r}}\frac{d\mathcal{R}_{2\pm}}{d\hat{r}} + \frac{\kappa_{1\pm}}{2}\frac{1}{\hat{r}}\frac{d^2\mathcal{R}_{2\pm}}{d\hat{r}^2}\right)\right\} \times\kappa_{3\pm}(\tau-\hat{t}) = 0,$$

where, clearly, the terms in each curly bracket are, or must be, independently equal to zero. This means that:

$$\frac{d\mathcal{R}_{1\pm}}{d\hat{r}} = \kappa_{4\pm}, \tag{77}$$

for some constant $\kappa_{4\pm}$. Therefore, on assuming (for simplicity and without loss of generality) that $\kappa_{3\pm} = \kappa_{4\pm}$ we get:

$$\mathcal{R}_{2\pm}\frac{d^2\mathcal{R}_{2\pm}}{d\hat{r}^2} + 2\frac{\kappa_{2\pm}}{\kappa_{1\pm}}\sqrt{\frac{2}{6\pm 2\sqrt{3}}}\left\{\frac{-\kappa_{3\pm}+\kappa_{1\pm}\kappa_{2\pm}}{\kappa_{1\pm}\kappa_{2\pm}}\right\}^{1/2}\hat{r}\left(\frac{\hat{r}}{\hat{r}+b}\right)^{1/2} = 0. \tag{78}$$

Now, equations (74), (77), and (78) necessarily imply that:

$$\mathcal{R}_{2\pm} = \kappa_{5\pm}\hat{r},$$

for some constant $\kappa_{5\pm}$ (and hence $\mathcal{R}_{1\pm} = \kappa_{1\pm}\kappa_{5\pm}$) and that $\kappa_{3\pm} = \kappa_{1\pm}\kappa_{2\pm}$. This means that $B_\pm(\hat{r},\hat{t}) = -\kappa_{3\pm}\kappa_{5\pm}$ and $C_\pm(\hat{r},\hat{t}) = \kappa_{3\pm}\kappa_{5\pm}\hat{r}(\tau-\hat{t})$ so that $F(\hat{r},\hat{t})$ is given by:

$$F_\pm(\hat{r},\hat{t}) = \kappa_{3\pm}\kappa_{5\pm}\exp\left\{-\kappa_{3\pm}\kappa_{5\pm}\hat{r}(\tau-\hat{t})\right\}. \tag{79}$$

Then, it is easy to show that, for $\kappa_{3\pm}\kappa_{5\pm} > 0$ (so that we have a bounded solution when $\hat{r}\to\infty$), b is necessarily equal to zero and $\kappa_{3\pm}\kappa_{5\pm}$ is given by:

$$\kappa_{3\pm}\kappa_{5\pm} = \frac{-1+\sqrt{1+2\left(6\pm 2\sqrt{3}\right)}}{6\pm 2\sqrt{3}}.$$

Thus, to leading order, the solution of the zero-coupon bond, $V(r,t)$, when the time to maturity is short, is given by:

$$V_{0\pm}(\hat{r},\hat{t}) = \frac{-1+\sqrt{1+2\left(6\pm 2\sqrt{3}\right)}}{6\pm 2\sqrt{3}}\exp\left\{-\frac{-1+\sqrt{1+2\left(6\pm 2\sqrt{3}\right)}}{6\pm 2\sqrt{3}}\hat{r}(\tau-\hat{t})\right\} + \left(1+\frac{1-\sqrt{1+2\left(6\pm 2\sqrt{3}\right)}}{6\pm 2\sqrt{3}}\right)\exp\left\{\pm\frac{1}{\sqrt{3}}\hat{r}(\tau-\hat{t})\right\}, \tag{80}$$

where $P_\pm(\hat{r},\hat{t})$ is a constant $P_{0\pm}$ given by $1+\left(1-\sqrt{1+2(6\pm 2\sqrt{3})}\right)/\left(6\pm 2\sqrt{3}\right)$ so that (80) satisfies the final condition that at maturity ($\hat{t}=\tau$), $V_{0\pm}(\hat{r},\hat{t}) = 1$. The boundary condition that as $\hat{r}\to\infty$, $V_{0\pm}(\hat{r},\hat{t})\to 0$, is satisfied automatically. The condition at $\hat{r}=0$ implies that (unsurprisingly) $V_{0\pm}(0,\hat{t}) = 1$.

(The dimensional version of equation (80) is simply:

$$V_{0\pm}(r,t) = \frac{-1+\sqrt{1+2\left(6\pm 2\sqrt{3}\right)}}{6\pm 2\sqrt{3}}V_T\exp\left\{-\frac{-1+\sqrt{1+2\left(6\pm 2\sqrt{3}\right)}}{6\pm 2\sqrt{3}}r(T-t)\right\} + \left(1+\frac{1-\sqrt{1+2\left(6\pm 2\sqrt{3}\right)}}{6\pm 2\sqrt{3}}\right)V_T\exp\left\{\pm\frac{1}{\sqrt{3}}r(T-t)\right\},$$

where, we recall, T and V_T are the known dimensional time to maturity and the known maturity value of the zero-coupon bond, respectively.)

6 Conclusion

We have performed an asymptotic analysis of the standard zero-coupon bond pricing equation in an environment where the sole source of uncertainty is the short-term interest rate r. (This analysis may be trivially extended to include coupon-bearing bonds.) The interest rate is governed by some stochastic differential equation. We have analysed the problem in two cases of interest: long-term-to-maturity and short-term-to-maturity bonds.

In the case of the long-term-to-maturity bond, to leading order, we observe that the non-dimensional r is deterministic. Hence, we solve the problem by assuming that r lies between two extreme values (which may, in general, be thought as absorbtion barriers) r_{min} and r_{max}. For the purposes of this study, these values are assumed to be known and, for simplicity, to be constants. (The idea of bounded r has been inspired by the work of Epstein & Wilmott (1998) and – as discussed earlier – by realistic interest rate evolution. It represents a departure from the standard assumption of bounded volatility in single-factor models.) We obtain, to leading order, a generalised closed-form solution in terms of the functions $V_{min}(r,t)$ and $V_{max}(r,t)$, where $V_{min}(r,t) \leq V(r,t) \leq V_{max}(r,t)$. (In general, the functions $V_{min}(r,t)$ and $V_{max}(r,t)$) are not known and must be determined by developing appropriate models; see, for example, Epstein &! Wilmott (1998). But, for the purposes of completion of this paper, we assume they are known or given.) Thus, our analysis, in this case, captures the desired characteristics of two eras of interest rate modelling: the classical random single-factor interest rate modelling and the recent non-probabilistics interest rate modelling. The developed method, however, may be applied – without too much difficulty – to many other complicated interest rate models.

For the short-term-to-maturity bond, to leading order, the non-dimensional r is described by a martingale process. We thus solve the problem by prescribing two conventionally accepted spatial boundary conditions as $r \to 0$ and as $r \to \infty$. But, at the same time, we make a radical departure from conventionality: we insist that the volatility should neither be arbitrarily prescribed nor be obtained from historical data, nor by fitting model's prices to the prices currently trading in the market, but should be found as part of solving the problem. The main reason for this departure is because the volatility – having been estimated by any of these other methods – seems never to stay the same; it appears to change in an unpredictable manner. We solve the two cases of interest: when the volatility is a constant parameter or an eigenvalue, and when the volatility is strictly a non-zero and a non-constant function of time. In both cases, the results are curious but inter! esting.

In the case of a constant volatility, we obtain a family of infinitely many bounded solutions which all lie in a narrow region. From a practical point of view, this solution may imply that the idea of one "correct" fair price for the zero-coupon bond is an ideal illusion. For the non-zero and non-constant function of time, the solution de-couples into two explicit functions (that all intersect at maturity, $t = T$) of which one may be viewed as the curve below which the bond is theoretically underpriced, and the other may be viewed as the curve above which the bond is theoretically overpriced, when the time to maturity is short. (Thus, from a practical point of view, this result may be an invaluable tool for a quick, but robust, pricing – to leading order – not only for short-term-to-maturity bonds but also for other fixed-income contracts which depend on the prices of the zero-coupon bonds when the time to maturity is short.) The question of whether these solutions can rea! lly be observed in practice is the question of linear stability analysis which is a subject of our future work.

Finally, in this paper we have particularly concentrated on the simple "single-factor" model for the zero-coupon bond pricing – other than on other existing more involved and complicated models – in the fixed-income world literature (where the movement of interest rates is acknowledged to be the major important factor in pricing complex instruments) is a call in Hull (2000): that single-factor models are still not yet masterfully understood, and are not as restrictive as it is often perceived. Besides, yield curve constraints which exist in the emerging markets, in particular, render more involved and complicated models impractical; moreover, in some asymptotic limits, many of these more involved and complicated models can be reduced to simple single-factor models.

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