Semiprimality relative to $\star$-congruences

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Abstract

We propose a notion of semiprimality for first order structures with respect to $\star$-congruences, and give a characterization which coincides with the classical case of algebras when the structures have no relation.[1]

1 Introduction

This work is an attempt to extend a result of Foster and Pixley in [4] to first order structures.

Let us recall some basic notions from the literature (cf. [2], [3] or [5]).

Definition 1.1 *Let E be a nonempty set, and $g : E^3 \to E$ be a ternary function.*
(i) g is a ***Malcev function*** *if $g(a, b, b) = a = g(b, b, a)$ for all $a, b \in E$.*
(ii) g is a ***majority function*** *if $g(a, a, b) = g(a, b, a) = g(b, a, a) = a$ for all $a, b \in E$.*
(iii) g is a ***Pixley function*** *if $g(a, a, b) = g(b, a, b) = g(b, a, a) = b$ for all $a, b \in E$.*
(iv) The ***discriminator function*** *on E is the ternary function $\boldsymbol{d}$ defined by*

$$\boldsymbol{d}(a, b, c) = \begin{cases} a & \text{if } a \neq b \ ; \\ c & \text{if } a = b. \end{cases}$$

Let $\mathcal{A} = (A; F^{\mathcal{A}})$ be an algebra; $\mathcal{A}$ is called *nontrivial* if A contains at least two elements.

A finite nontrivial algebra $\mathcal{A}$ is called *semiprimal* if every n-ary function $f : A^n \to A$, $n \geq 1$, which preserves the subuniverses of $\mathcal{A}$ is a term function. A *semiprimal* algebra $\mathcal{A}$ is characterized by the following two facts (cf. [4]):
- $(SP1)$ The discriminator function on A is a term function of $\mathcal{A}$, and
- $(SP2)$ Distinct nontrivial subalgebras of $\mathcal{A}$ are nonisomorphic, and no subalgebra of $\mathcal{A}$ has a proper automorphism.

From $(SP1)$, it can be shown that any subuniverse E of $\mathcal{A}^2$ is either the product of two subuniverses of $\mathcal{A}$ or (the graph of) an isomorphism between two subalgebras of $\mathcal{A}$. So with $(SP2)$, any such E is the product of two subuniverses or the identity map of a subalgebra of $\mathcal{A}$.

Now, a lemma of Baker and Pixley states that when $\mathcal{A}$ has a majority term, a function $h : A^n \to A$, $n \geq 1$, is a term function iff h preserves the subuniverses of $\mathcal{A}^2$ (cf [1], or a proof in [2]). This will be our main tool, noting that $A^2 = \nabla_A$ is the largest congruence of $\mathcal{A}$, and in the case of a semiprimal $\mathcal{A}$, h preserves the subuniverses of $\mathcal{A}^2$ iff it preserves the subuniverses of $\mathcal{A}$.

Our motivation comes from a work of N. Weaver ([8]) on some classes of first order structures.

Definition 1.2 *([8]) Let $\mathfrak{A} = (A; F^{\mathfrak{A}}; R^{\mathfrak{A}})$ and $\mathfrak{B} = (B; F^{\mathfrak{B}}; R^{\mathfrak{B}})$ be first order structures of the same type.*

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(i) A morphism $\lambda : \mathfrak{A} \to \mathfrak{B}$ is called a ***$\star$-morphism*** *if for any m-ary $r \in R$, and $a_1, \ldots, a_m \in A$, $\langle a_1, \ldots, a_m \rangle \in r^{\mathfrak{A}}$ iff $\langle \lambda(a_1), \ldots, \lambda(a_m) \rangle \in r^{\mathfrak{B}}$. In this case, the substructure $\lambda(\mathfrak{A})$ of $\mathfrak{B}$ is called a* ***$\star$-image*** *of $\mathfrak{A}$.*
(ii) A congruence θ of $\mathfrak{A}$ is called a ***$\star$-congruence*** *if it is compatible with the relations of $\mathfrak{A}$; that is, for any m-ary r in R and $\langle u_i, v_i \rangle \in \theta$ for $1 \leq i \leq m$, $\vec{u} \in r^{\mathfrak{A}}$ iff $\vec{v} \in r^{\mathfrak{A}}$.*
(iii) A class $\mathcal{K}$ of structures of the same type is called a ***$\star$-variety*** *if it is closed under products, substructures and $\star$-images.*

So, $\star$-congruences are exactly the kernels of $\star$-morphisms.

The set $Con_\star(\mathfrak{A})$ of $\star$-congruences of $\mathfrak{A}$ is a sublattice of $Con(\mathfrak{A})$; in fact $Con_\star(\mathfrak{A})$ is a complete lattice with smallest element $id_A = \triangle_A$, and largest element denoted by $1_{\mathfrak{A}}$; in general $1_{\mathfrak{A}} \neq A^2 = \nabla_A$.

Using $\star$-congruences, N. Weaver established some Malcev like conditions, and a structure theorem for $\star$-varieties. Our aim is to follow this line and formulate a notion of semiprimality for a first order structure $\mathfrak{A} = (A; F^{\mathfrak{A}}; R^{\mathfrak{A}})$ where ∇_A is replaced by $1_{\mathfrak{A}}$.

For this we need to adapt some definitions on algebras to the situation of first order structures we want to investigate.

Notation 1

Let $\mathfrak{A} = (A; F^{\mathfrak{A}}; R^{\mathfrak{A}})$ be a first order structure.

(i) Let θ be an equivalence relation on A and a be an element of A; then $[a]_\theta$ is the θ class of a. When $\theta = 1_{\mathfrak{A}}$, we simply denote $[a]_{1_{\mathfrak{A}}}$ by $\overline{a}$.

(ii) For a subset X of $A, Sg(X)$ denotes the subuniverse of $\mathfrak{A}$ generated by X; if $X = \{x_1, \ldots, x_n\}$, we simply denote $Sg(X)$ by $Sg(x_1, ..., x_n)$.

(iii) Given two elements a and b of $\mathfrak{A}^n$, let $H(a, b)$ denote the subuniverse of $\mathfrak{A}^2$ defined by $H(a, b) = Sg(\langle a(1), b(1) \rangle, \ldots, \langle a(n), b(n) \rangle)$.

In section 2 we give a version of the Baker-Pixley Lemma on term representation of functions, and a weak Malcev like condition for $\star$-congruence permutability. Section 3 is devoted to formulating and characterizing a notion of semiprimality for $\mathfrak{A}$.

2 Preliminaries

Malcev like conditions given by N. Weaver state that a $\star$-variety $\mathcal{V}$ is $\star$-congruence permutable (respectively, arithmetical) iff there is a ternary term t such that for each structure $\mathfrak{A} \in \mathcal{V}$ and each $a \in A$, $t^{\mathfrak{A}}$ is a Malcev function (respectively, a Pixley function) on $\overline{a} = [a]_{1_{\mathfrak{A}}}$.

In this section, for a finite structure $\mathfrak{A} = (A; F^{\mathfrak{A}}; R^{\mathfrak{A}})$, we give a version of the Baker-Pixley Lemma on term representation of functions, where $1_{\mathfrak{A}}$ is substituted to $A^2 = \nabla_A$. Then we examine some conditions which force the substructures of $\mathfrak{A}$ to be $\star$-congruence permutable.

Definition 2.1 *Let $h : A^n \to A$, $n \geq 1$, be a function.*
(i) h is said to be ***termal on classes*** *if for each element $a \in A$, there is an n-ary term t such that $t^{\mathfrak{A}}$ and h coincide on $([a]_{1_{\mathfrak{A}}})^n = \overline{a}^n$.*
(ii) h is said to be ***weakly termal*** *if for any subset E of A^n satisfying property*

$$(P): \qquad a, b \in E \text{ implies } \langle a(i), b(i) \rangle \in 1_{\mathfrak{A}} \text{ for } 1 \leq i \leq n,$$

there is an n-ary term t such that $t^{\mathfrak{A}}$ and h coincide on E.
(iii) An n-ary term t ***represents*** *h* ***on classes*** *if for each $a \in A$, $t^{\mathfrak{A}}$ and h coincide on $\overline{a}^n$. In this case we say that h is* ***term representable on classes*** *by t.*

A weakly termal function $h : A^n \to A$ is compatible with $1_{\mathfrak{A}}$, that is,

$$\langle a_i, b_i\rangle \in 1_{\mathfrak{A}} \text{ for } 1 \leq i \leq n \text{ implies } \langle h(\vec{a}), h(\vec{b})\rangle \in 1_{\mathfrak{A}}.$$

Remark 2.1

We note that:
(i) *Weakly termal* functions and functions which are *term representable on classes* are *termal on classes.*
(ii) For a unary function h, *weakly termal* and *termal on classes* are equivalent, and h is *term representable on classes* means h is a *term function.*
(iii) A constant function h is *termal on classes* iff it is a *term function.*

We will use the following version of the Baker-Pixley Lemma.

Lemma 2.1 *Suppose that* $\mathfrak{A}$ *has a majority function which is termal on classes, and let* $h : A^n \to A$, $n \geq 1$, *be a function.*
(i) The following conditions are equivalent for an element $u \in A$.
(i_1) *For any elements* $e_1, e_2, \ldots, e_n$ *of* $\overline{u}^2$, *the element* $h^{\mathfrak{A}^2}(e_1, \ldots, e_n)$ *belongs to* $Sg(e_1, \ldots, e_n)$ *in* $\mathfrak{A}^2$.
(i_2) *For any finite subset* U *of* $\overline{u}^n$ *there is an* n*-ary term* t *such that* $t^{\mathfrak{A}}$ *and* h *coincide on* U.
(ii) h *is weakly termal if and only if* h *preserves the subuniverses of* $1_{\mathfrak{A}}$.

Proof (i): Clearly, we only have to prove that (i_1) implies (i_2). Let $a, b \in \overline{u}^n$; then $\langle a(1), b(1)\rangle, \ldots, \langle a(n), b(n)\rangle$ are elements of $\overline{u}^2$. By hypothesis, the element $h^{\mathfrak{A}^2}(\langle a(1), b(1)\rangle, \ldots, \langle a(n), b(n)\rangle)$ belongs to $Sg(\langle a(1), b(1)\rangle, \ldots, \langle a(n), b(n)\rangle)$; so we have $h^{\mathfrak{A}^2}(\langle a(1), b(1)\rangle, \ldots, \langle a(n), b(n)\rangle =$ $t^{\mathfrak{A}^2}(\langle a(1), b(1)\rangle, ..., \langle a(n), b(n)\rangle)$ for some n-ary term t; that is $h(a(1), ..., a(n)) = t^{\mathfrak{A}}(a(1), \ldots, a(n))$ and $h(b(1), \ldots, b(n)) = t^{\mathfrak{A}}(b(1), \ldots, b(n))$.

Suppose that for any subset E_0 of U with $2 \leq card(E_0) \leq k$, there is an n-ary term which coincides with h on E_0. If $card(U) > k$, let $E_1 \subseteq U$ with $card(E_1) = k + 1$; choose three distinct elements $e_1, e_2, e_3 \in E_1$. Then there are terms t_1, t_2, t_3 such that $t_i^{\mathfrak{A}}$ coincides with h on $E_1 \backslash \{e_i\}$ for $1 \leq i \leq 3$.

Let $e \in E_1$; then $e \neq e_2$ implies $\langle h(e), h(e_1)\rangle = \langle t_2^{\mathfrak{A}}(e), t_2^{\mathfrak{A}}(e_1)\rangle \in 1_{\mathfrak{A}}$, and $e \neq e_3$ implies $\langle h(e), h(e_1)\rangle = \langle t_3^{\mathfrak{A}}(e), t_3^{\mathfrak{A}}(e_1)\rangle \in 1_{\mathfrak{A}}$. So, if $v = h(e_1)$, then $h(E_1) \subseteq \overline{v}$ and $t_i^{\mathfrak{A}}(E_1) \subseteq \overline{v}$ for $1 \leq i \leq 3$.

Now let t_4 be a ternary term interpolating the majority function on $\overline{v}$, and consider the term $\sigma(x_1, ..., x_n) = t_4(t_1(\vec{x}), t_2(\vec{x}), t_3(\vec{x}))$. Then h coincides with at least two of the $t_i^{\mathfrak{A}}, 1 \leq i \leq 3$, on each element of E_1, so that $\sigma^{\mathfrak{A}}(e) = h(e)$ for each $e \in E_1$. Since U is finite, we can iterate the process and obtain a term which coincides with h on U.

(ii): The proof is essentially the same, using property (P) in Definition 2.1, and can be found in [6]. □

Let $\mathfrak{B}$ and $\mathfrak{D}$ be substructures of $\mathfrak{A}$, θ be a congruence of $\mathfrak{B}$, φ be a congruence of $\mathfrak{D}$, and $\alpha : \mathfrak{B}/\theta \to \mathfrak{D}/\varphi$ be an isomorphism.

The set $E = \{\langle b, d\rangle \in B \times D; \alpha(b/\theta) = d/\varphi\}$ is called the *lifting* of α. When $E \subseteq 1_{\mathfrak{A}}$, we say that α is a $1_{\mathfrak{A}}$ *isomorphism*; in this case, $\theta \subseteq 1_{\mathfrak{A}} \cap B^2$ and similarly for φ, so that θ is a $\star$-congruence of $\mathfrak{B}$ and φ is a $\star$-congruence of $\mathfrak{D}$.

Remark 2.2

Let $\lambda : \mathfrak{A} \to \mathfrak{B}$ be an epimorphism.
(i) For any congruence φ of $\mathfrak{B}$, $\lambda^{-1}(\varphi) = \{\langle u, v\rangle \in A^2 : \langle\lambda(u), \lambda(v)\rangle \in \varphi\}$ is a congruence of $\mathfrak{A}$; and if θ is a congruence of $\mathfrak{A}$, then $\lambda(\theta) = \{\langle\lambda(u), \lambda(v)\rangle : \langle u, v\rangle \in \theta\}$ is a congruence of $\mathfrak{B}$ if the kernel of λ is contained in θ.
(ii) If φ, ψ are congruences of $\mathfrak{B}$, one easily verify that:

(ii_1) $\lambda(\lambda^{-1}(\varphi)\cap\lambda^{-1}(\psi))=\lambda(\lambda^{-1}(\varphi))\cap\lambda(\lambda^{-1}(\psi))=\varphi\cap\psi$.

(ii_2) $\lambda(\lambda^{-1}(\varphi)\circ\lambda^{-1}(\psi))=\lambda(\lambda^{-1}(\varphi))\circ\lambda(\lambda^{-1}(\psi))=\varphi\circ\psi$; and by iteration we have $\lambda(\lambda^{-1}(\varphi)\vee\lambda^{-1}(\psi))=\varphi\vee\psi$.

This implies that if θ and τ are congruences of $\mathfrak{A}$ containing the kernel of λ, then $\lambda(\theta\cap\tau)=\lambda(\theta)\cap\lambda(\tau)$ and $\lambda(\theta\vee\tau)=\lambda(\tau)\vee\lambda(\tau)$.

Lemma 2.2 *Let $\lambda:\mathfrak{A}\to\mathfrak{B}$ be a $\star$-epimorphism.*
(i) For any $\star$-congruence φ of $\mathfrak{B}$, $\lambda^{-1}(\varphi)$ is a $\star$-congruence of $\mathfrak{A}$.
(ii) If θ is a $\star$-congruence of $\mathfrak{A}$ containing the kernel of λ, then $\lambda(\theta)$ is a $\star$-congruence of $\mathfrak{B}$.
(iii) If $Con_\star(\mathfrak{A})$ is permutable (respectively, distributive) then so is $Con_\star(\mathfrak{B})$.

Proof. (i): Consider an m-ary $r\in R$, and $\langle x_i,y_i\rangle\in\lambda^{-1}(\varphi)$ for $1\leq i\leq m$ such that $\langle x_1,\ldots,1_m\rangle\in r^{\mathfrak{A}}$; then $\langle\lambda(x_i),\lambda(y_i)\rangle\in\varphi$ for each i and $\langle\lambda(x_1),\ldots,\lambda(x_m)\rangle\in r^{\mathfrak{B}}$; so $\langle\lambda(y_1),\ldots,\lambda(y_m)\rangle\in r^{\mathfrak{B}}$ as φ is a $\star$-congruence. But then $\langle y_1,...,y_m\rangle\in r^{\mathfrak{A}}$ since λ is a $\star$-morphism. This show that $\lambda^{-1}(\varphi)$ is a $\star$-congruence of $\mathfrak{A}$.

(ii): The proof is similar to that of (i), noting that the kernel of λ is a subset of θ.

(iii): Let φ,ψ and η be $\star$-congruences of $\mathfrak{B}$.

Suppose that $Con_\star(\mathfrak{A})$ is permutable; then
$\varphi\circ\psi=\lambda(\lambda^{-1}(\varphi)\circ\lambda^{-1}(\psi))=\lambda(\lambda^{-1}(\psi)\circ\lambda^{-1}(\varphi))=\psi\circ\varphi$.

Suppose that $Con_\star(\mathfrak{A})$ is distributive; then,

$$\begin{aligned}\varphi\cap(\psi\vee\eta)&=\lambda(\lambda^{-1}(\varphi))\cap\lambda(\lambda^{-1}(\psi)\vee\lambda^{-1}(\eta))\\&=\lambda(\lambda^{-1}(\varphi)\cap(\lambda^{-1}(\psi)\vee\lambda^{-1}(\eta)))\\&=\lambda((\lambda^{-1}(\varphi)\cap\lambda^{-1}(\psi))\vee(\lambda^{-1}(\varphi)\cap\lambda^{-1}(\eta)))\\&=\lambda(\lambda^{-1}(\varphi)\cap\lambda^{-1}(\psi))\vee\lambda(\lambda^{-1}(\varphi)\cap\lambda^{-1}(\eta))\\&=(\varphi\cap\psi)\vee(\varphi\cap\eta)).\end{aligned}$$

□

From this lemma, we see that the $\star$-congruences of $\mathfrak{B}$ are images by λ of the $\star$-congruences of $\mathfrak{A}$; in particular, $1_{\mathfrak{B}}=\lambda(1_{\mathfrak{A}})$.

Definition 2.2 *A subdirect product of structures $\mathfrak{B}\subseteq\prod_{1\leq i\leq n}\mathfrak{B}_i$ is said to be $\star$-**subdirect** if every projection map $p_i:\mathfrak{B}\to\mathfrak{B}_i$ is a $\star$-morphism.*

Lemma 2.3 *Let $\mathfrak{M}\subseteq\mathfrak{B}\times\mathfrak{D}$ be a $\star$-subdirect product of structures such that the kernels of the projection maps $p_1:\mathfrak{M}\to\mathfrak{B}$ and $p_2:\mathfrak{M}\to\mathfrak{D}$ are permuting congruences. Then there are $\star$-epimorphisms $\beta:\mathfrak{B}\to\mathfrak{N}$ and $\delta:\mathfrak{D}\to\mathfrak{N}$ such that $M=\{\langle b,d\rangle\in B\times D;\beta(b)=\delta(d)\}$.*

Proof Since $\mathfrak{M}\subseteq\mathfrak{B}\times\mathfrak{D}$ is a $\star$-subdirect product, the projection maps $p_1:\mathfrak{M}\to\mathfrak{B}$ and $p_2:\mathfrak{M}\to\mathfrak{D}$ are $\star$-epimorphisms; then $\rho_i:=ker(p_i)$ is an element of $Con_\star(\mathfrak{M})$ for $i=1,2$, and so is $\rho=\rho_1\vee\rho_2=\rho_1\circ\rho_2$.

Let p: $\mathfrak{M}\to\mathfrak{M}/\rho$ be the canonical $\star$-epimorphism; there are $\star$-epimor-phisms h_1: $\mathfrak{B}\to\mathfrak{M}/\rho$ and $h_2:\mathfrak{D}\to\mathfrak{M}/\rho$ such that $p=h_i\circ p_i$ for each i.

If $\langle b,d\rangle\in M$, then $h_1(b)=(h_1\circ p_1)(\langle b,d\rangle)=(h_2\circ p_2)(\langle b,d\rangle)=h_2(d)$.

Conversely, let $\langle b,d\rangle\in B\times D$ such that $h_1(b)=h_2(d)$; there are elements $u\in D$ and $v\in B$ such that $\langle b,u\rangle\in M$ and $\langle v,d\rangle\in M$. Then $p(\langle b,u\rangle)=h_1(b)=h_2(d)=p(\langle v,d\rangle)$, and $\langle\langle v,d\rangle,\langle b,u\rangle\rangle$ belongs to $\rho=\rho_1\circ\rho_2$. Let $\langle e_1,e_2\rangle\in M$ such that $\langle v,d\rangle\rho_2\langle e_1,e_2\rangle\rho_1\langle b,u\rangle$; then $d=e_2$ and $e_1=b$, so that $\langle b,d\rangle=\langle e_1,e_2\rangle\in M$. □

In the proof of the lemma, let $\theta:=ker(h_1)$ and $\varphi:=ker(h_2)$, and consider the canonical isomorphisms $\tilde{h}_1:\mathfrak{B}/\theta\to\mathfrak{N}$ and $\tilde{h}_2:\mathfrak{D}/\varphi\to\mathfrak{N}$. Then, M is the lifting of the isomorphism $\alpha=\tilde{h}_2^{-1}\circ\tilde{h}_1:\mathfrak{B}/\theta\to\mathfrak{D}/\varphi$.

Definition 2.3 *$1_{\mathfrak{A}}$ is said to be **hereditarily maximal** if for every substructure $\mathfrak{B}$ of $\mathfrak{A}$, $1_{\mathfrak{B}} = 1_{\mathfrak{A}} \cap B^2$.*

Proposition 2.4 *Let $1_{\mathfrak{A}}$ be hereditarily maximal and $\mathfrak{M}$ be a substructure of $\mathfrak{A}^2$ with $M \subseteq 1_{\mathfrak{A}}$. If $\mathfrak{A}$ has a Malcev function (respectively, a Pixley function) which is termal on classes, then $Con_\star(\mathfrak{M})$ is permutable (respectively, arithmetical).*

Proof Let θ, φ and ψ be $\star$-congruences of $\mathfrak{M}$.

For permutability, let $\langle a, b\rangle \in \theta \circ \varphi$ and $c \in M$ such that $a\varphi c\theta b$. Since M is a subuniverse of $1_{\mathfrak{A}}$, the two projections of M are $\star$-morphisms; so by Lemma 2.2 and the maximal heredity of $1_{\mathfrak{A}}$, if $a = \langle a_1, a_2\rangle, b = \langle b_1, b_2\rangle$ and $c = \langle c_1, c_2\rangle$, then a_i, b_i, c_i are elements of $[a_1]_{1_{\mathfrak{A}}}$ for $1 \leq i \leq 2$. Let t be a ternary term representing the given Malcev function on $[a_1]_{1_{\mathfrak{A}}}$; then

$a = t^{\mathfrak{A}^2}(a, b, b)\theta t^{\mathfrak{A}^2}(a, c, b)\varphi t^{\mathfrak{A}^2}(c, c, b) = b$, and $\langle a, b\rangle \in \varphi \circ \theta$.

For arithmeticity let $\langle a, b\rangle \in \theta \cap (\varphi \circ \psi)$; then $a\psi c\varphi b$ for some $c \in M$. Let $a = \langle a_1, a_2\rangle, b = \langle b_1, b_2\rangle$ and $c = \langle c_1, c_2\rangle$; then $a_i, b_i, c_i \in [a_1]_{1_{\mathfrak{A}}}$ for $1 \leq i \leq 2$.
Let t be a term representing the given Pixley function on $[a_1]_{1_{\mathfrak{A}}}$; then

$t^{\mathfrak{A}^2}(a, c, b)\theta t^{\mathfrak{A}^2}(a, c, a) = a$ and $t^{\mathfrak{A}^2}(a, c, b)\theta t^{\mathfrak{A}^2}(b, c, b) = b$. So

$a = t^{\mathfrak{A}^2}(a, b, b)(\theta \wedge \varphi)t^{\mathfrak{A}^2}(a, c, b)(\theta \wedge \psi)t^{\mathfrak{A}^2}(a, a, b) = b$, and

$\langle a, b\rangle \in (\theta \wedge \psi) \circ (\theta \wedge \varphi)$. Thus $\theta \wedge (\varphi \circ \psi) \subseteq (\theta \wedge \psi) \circ (\theta \wedge \varphi)$; the reverse inclusion is obvious, and we have the equality. When $\theta = 1_{\mathfrak{M}}$ we obtain $\varphi \circ \psi = \psi \circ \varphi$: thus $\theta \wedge (\varphi \circ \psi) = (\theta \wedge \psi) \circ (\theta \wedge \varphi)$, and $Con_\star(\mathfrak{M})$ is arithmetical. □

So, if $\mathfrak{A}$ has a Malcev function which is termal on classes and $1_{\mathfrak{A}}$ is hereditarily maximal, then

- every substructure of $\mathfrak{A}$ is $\star$-congruence permutable, and
- by Lemma 2.3 every subuniverse of $1_{\mathfrak{A}}$ is the lifting of a $1_{\mathfrak{A}}$ isomorphism.

Let $a_1, \ldots, a_m$ be elements of A^n, and $a^i := \langle a_1(i), ..., a_m(i)\rangle$ for $1 \leq i \leq n$; consider the congruence $Cong(a_k, a_l) := \bigvee\{\theta(\langle a_k(\alpha), a_l(\alpha)\rangle); 1 \leq \alpha \leq n\}$ of $\mathfrak{A}$, and the subuniverse

$B(a^1, \ldots, a^n) := \{x \in A^m; \langle x(k), x(l)\rangle \in Cong(a_k, a_l)$ for $1 \leq k, l \leq m\}$ of $\mathfrak{A}^m$; then $Sg(a^1, \ldots, a^n) \subseteq B(a^1, \ldots, a^n)$.

We will use the following definition and result in the next section.

Definition 2.4 ([7]) *Let $\mathfrak{A}$ be a finite nontrivial structure.*
*(i) $\mathfrak{A}$ is called $\star$-**primal** if every n-ary function $f : A^n \to A$, $n \geq 1$, which is $1_{\mathfrak{A}}$ compatible is term representable on classes.*
*(ii) $\mathfrak{A}$ is called **weakly $\star$-primal** if every such function is termal on classes.*

A structure $\mathfrak{A}$ is called *minimal* if it has no proper substructure, and $\mathfrak{A}$ is called *rigid* if it has no proper automorphism (i.e., id_A is the only automorphism of $\mathfrak{A}$).

Theorem 2.5 ([7]) *Let $\mathfrak{A}$ be a finite nontrivial structure.*
(i) $\mathfrak{A}$ is weakly $\star$-primal iff the following properties are satisfied:

*(i_1). $\mathfrak{A}$ is rigid, and **d** is termal on classes.*

(i_2). $\triangle_A$ and $1_{\mathfrak{A}}$ are the only subuniverses of $1_{\mathfrak{A}}$.

(ii) $\mathfrak{A}$ is $\star$-primal iff the following conditions are satisfied:

(ii_1). The only subuniverses of $\mathfrak{A}^2$ are $\triangle_A$, $1_{\mathfrak{A}}$ and $A^2 = \bigtriangledown_A$.

(ii_2). For any non zero natural numbers m and n, and elements $a_1, \ldots, a_m$ of $\bigcup_{u \in A} \overline{u}^n$, if $a^i := \langle a_1(i), \ldots, a_m(i)\rangle$ for $1 \leq i \leq n$ then $B(a^1, \ldots, a^n) = Sg(a^1, \ldots, a^n)$ as subuniverses of $\mathfrak{A}^m$. □

Condition (ii_1) in this theorem implies that $\mathfrak{A}$ is minimal and rigid. Moreover, when $\mathfrak{A}$ is $\star$-primal, the discriminator function **d** on A is term representable on classes.

3 $\star$-semiprimality

From the characterization of a semiprimal algebra (i.e., $(SP1)$ and $(SP2)$ in the Introduction), a similar notion for first order structures is expected to imply that the discriminator function $\mathbf{d}$ is term representable on classes. What we propose in this section does fulfill this requirement, and also takes care of $(SP2)$.

When $\mathbf{d}$ is a term function for an algebra $\mathcal{A} = (A; F^{\mathcal{A}})$, $\mathcal{A}$ and its subalgebras are simple. The next definition gives a notion of simplicity with respect to $\star$-congruences on first order structures

Definition 3.1
(i) A $\star$-congruence θ of $\mathfrak{A}$ is said to be ***simple*** *if for each element $a \in A$, $[a]_\theta = \{a\}$ or $[a]_\theta = [a]_{1_{\mathfrak{A}}}$.*
$\mathfrak{A}$ is said to be ***$\star$-simple*** *if every $\star$-congruence of $\mathfrak{A}$ is simple.*
(ii) $\mathfrak{A}$ is said to be ***hereditarily $\star$-simple*** *if every substructure of $\mathfrak{A}$ is $\star-$simple.*

In [6] it is shown that when $\mathbf{d}$ is termal on classes, $\mathfrak{A}$ is $\star$-simple and $Con_\star(\mathfrak{A})$ is arithmetical. If moreover $1_{\mathfrak{A}}$ is hereditarily maximal, then $\mathfrak{A}$ is hereditarily $\star$-simple, and since $\mathbf{d}$ is a Malcev function, every subuniverse E of $1_{\mathfrak{A}}$ is the lifting of some $1_{\mathfrak{A}}$ isomorphism $\alpha : \mathfrak{B}/\theta \to \mathfrak{D}/\varphi$, where θ and φ are simple $\star$-congruences of $\mathfrak{B}$ and $\mathfrak{D}$ respectively.

E and α may not be identity maps, but we want them to be as close as possible to the identity.

Definition 3.2 *Let E be the lifting of the $1_{\mathfrak{A}}$ isomorphism $\alpha : \mathfrak{B}/\theta \to \mathfrak{D}/\varphi$; we say that:*
(i) E is a ***simple*** *$1_{\mathfrak{A}}$* ***lifting*** *if θ and φ are simple $\star$-congruences.*
(ii) α is a $1_{\mathfrak{A}}$ ***semi-identity*** *if for every $\langle b, d\rangle \in E$, $b \neq d$ implies $b/\theta = \bar{b} \cap B$.*

If E is the lifting of a $1_{\mathfrak{A}}$ semi-identity and $\langle b, d\rangle \in E$, then $b/\theta \neq \{b\}$ implies $b/\theta = \bar{b} \cap B$; so θ and φ are simple $\star$-congruences.

Lemma 3.1 *Let $\mathfrak{A}$ be a structure such that $1_{\mathfrak{A}}$ is hereditarily maximal and every $1_{\mathfrak{A}}$ compatible function $h : A^n \to A$, $n \geq 1$, which preserves the subuniverses of $\mathfrak{A}$ is termal on classes; then:*
(i) The discriminator function $\mathbf{d}$ *is termal on classes.*
(ii) Every subuniverse of $1_{\mathfrak{A}}$ is the lifting of a $1_{\mathfrak{A}}$ semi-identity.

Proof (i): Consider the function $\mathbf{q} : A^3 \to A$ defined by

$$\mathbf{q}(a, b, c) = \begin{cases} c & \text{if } a, b \text{ and } c \text{ are in the same } 1_{\mathfrak{A}} \text{ class and } a = b; \\ a & \text{if } a, b \text{ and } c \text{ are in the same } 1_{\mathfrak{A}} \text{ class and } a \neq b; \\ b & \text{elsewhere.} \end{cases}$$

Then $\mathbf{q}$ is $1_{\mathfrak{A}}$ compatible and preserves the subuniverses of $\mathfrak{A}$. Thus $\mathbf{q}$ is termal on classes. Since $\mathbf{q}$ coincide with $\mathbf{d}$ on classes, $\mathbf{d}$ is also termal on classes.

(ii): Let E be a subuniverse of $1_{\mathfrak{A}}$; since $\mathbf{d}$ is a Malcev function and is termal on classes, E is the lifting of some $1_{\mathfrak{A}}$ isomorphism $\alpha : \mathfrak{B}/\theta \to \mathfrak{D}/\varphi$, where θ and φ are simple $\star$-congruences. Let $\langle b, d\rangle \in E$.

Claim 1: If $b \notin D$, then $b/\theta = \bar{b} \cap B$.
Proof : Suppose that b does not belong to D and $b/\theta \neq \bar{b} \cap B$; there is some element $c \in \bar{b} \cap B$ such that $c/\theta \neq b/\theta$; then $d/\varphi = \alpha(b/\theta) \neq \alpha(c/\theta) = e/\varphi$.
Define the function $h : A^2 \to A$ by

$$h(x, y) = \begin{cases} y & \text{if } x \text{ and } y \text{ are in the same } 1_{\mathfrak{A}} \text{ class and } x = b; \\ x & \text{elsewhere.} \end{cases}$$

Then h is $1_{\mathfrak{A}}$ compatible and preserves the subuniverses of $\mathfrak{A}$. So h is termal on classes; $\alpha(h^{\theta}(b/\theta, c/\theta)) = h^{\varphi}(\alpha(b/\theta), \alpha(c/\theta))$; that is $\alpha(c/\theta) = h^{\varphi}(d/\varphi, e/\varphi)$, and we would have $e/\varphi = d/\varphi$, a contradiction.

Claim 2: $b \in D$ implies $\alpha(b/\theta) = b/\varphi$.

Proof : If $b \in D$ and $\alpha(b/\theta) \neq b/\varphi$, then $\alpha(b/\theta) = d/\varphi \neq b/\varphi$, and $b/\varphi = \alpha(u/\theta)$ for some $u \in \overline{b} \cap B$ with $b/\theta \neq u/\theta$.

The function h defined in the proof of Claim 1 is termal on classes. So, $\alpha(h(^{\theta}(b/\theta, u/\theta)) = h^{\varphi}(\alpha(b/\theta), \alpha(u/\theta))$; that is $\alpha(u/\theta) = h^{\varphi}(d/\varphi, b/\varphi) = d/\varphi$, a contradiction.

Claim 3: $b/\theta \subseteq D$ implies $b/\theta = B \cap (b/\varphi)$.

Proof : Suppose that $b/\theta \neq \{b\}$ or $b/\varphi \neq \{b\}$, then $b/\theta = \overline{b} \cap B$ or $b/\varphi = \overline{b} \cap D$ since θ and φ are simple $\star$-congruences. But α is a $1_{\mathfrak{A}}$ isomorphism, and $\alpha(b/\theta) = b/\varphi$; so $b/\theta = \overline{b} \cap B$ iff $b/\varphi = \overline{b} \cap D$; thus $\overline{b} \cap B = b/\theta \subseteq \overline{b} \cap D = b/\varphi$.

To complete the proof of the theorem, suppose that $b \neq d$.

· If $b \notin D$, then $b/\theta = \overline{b} \cap B$ by Claim 1.

· If $b \in D$, then $b/\varphi = \alpha(b/\theta) = d/\varphi$; so $b/\varphi = d/\varphi = \overline{d} \cap D$, since φ is a simple $\star$-congruence. For any element $e \in \overline{b} \cap B$ we have $\alpha(e/\theta) \subseteq \overline{e} \cap D = \overline{d} \cap D = b/\varphi = \alpha(b/\theta)$; so $e/\theta = b/\theta$, and $b/\theta = \overline{b} \cap B$. □

Remark 3.1

With the hypothesis of this lemma, we have the following facts from the above proof:

Fact-1 . Let $\mathfrak{B}$ and $\mathfrak{D}$ be substructures of $\mathfrak{A}$, θ be a congruence of $\mathfrak{B}$ and φ be a congruence of $\mathfrak{D}$. Then,

(i) There exists at most one $1_{\mathfrak{A}}$ isomorphism $\alpha : \mathfrak{B}/\theta \rightarrow \mathfrak{D}/\varphi$.

(ii) If such an isomorphism exists it must be a $1_{\mathfrak{A}}$ semi-identity. Moreover if $\mathfrak{B} \subseteq \mathfrak{D}$, then $\theta = \varphi \cap B^2$, the restriction of φ to $\mathfrak{B}$; in particular if $\mathfrak{B} = \mathfrak{D}$, then $\theta = \varphi$ and α is the identity.

Fact-2 . If τ is a (simple) $\star$-congruence of $\mathfrak{A}$, there is no $1_{\mathfrak{A}}$ isomorphism between two distinct non-trivial substructures of $\mathfrak{A}/\tau$.

This suggests the definition we give below. For the rest of the section, $\mathfrak{A}$ is a finite nontrivial structure.

Definition 3.3

*(i) $\mathfrak{A}$ is said to be **weakly $\star$-semiprimal** if every $1_{\mathfrak{A}}$ compatible function which preserves the subuniverses of $\mathfrak{A}$ is termal on classes.*

*(ii) $\mathfrak{A}$ is said to be **$\star$-semiprimal** if every $1_{\mathfrak{A}}$ compatible function which preserves the subuniverses of $\mathfrak{A}$ is term representable on classes.*

Corollary 3.2

(i) $\mathfrak{A}$ is weakly $\star$-primal iff $\mathfrak{A}$ is weakly $\star$-semiprimal and minimal.

(ii) $\mathfrak{A}$ is $\star$-primal iff $\mathfrak{A}$ is $\star$-semiprimal and minimal. □

We introduced the notion of a minimal structure after Definition 2.4.

Theorem 3.3 *$\mathfrak{A}$ is weakly $\star$-semiprimal iff the following conditions are satisfied.*

(i) The discriminator function $\boldsymbol{d}$ is termal on classes.

(ii) For any non zero natural number n, any element $a \in A$ and distinct elements $b, c \in \overline{a}^n$, if $B = Sg(b(1), \ldots, b(n))$ and $C = Sg(c(1), \ldots, c(n))$, then $H(b, c) = (B \times C) \cap 1_{\mathfrak{A}}$.

Proof ($\Rightarrow$): (i) follows from the proof of Lemma 3.1.(i).

For (ii), we have $H(b,c) = Sg(\langle b(1), c(1)\rangle, \ldots, \langle b(n), c(n)\rangle) \subseteq (B \times C) \cap 1_{\mathfrak{A}}$. Now, let $\langle u, v\rangle \in (B \times C) \cap 1_{\mathfrak{A}}$; there are n-ary terms t_1 and t_2 such that $u = t_1^{\mathfrak{A}}(b)$ and $v = t_2^{\mathfrak{A}}(c)$. Consider the function $g : A^n \to A$ defined by

$$g(x) = \begin{cases} t_1^{\mathfrak{A}}(x) & \text{if } x \in (\overline{a}^n \setminus \{c\}); \\ t_2^{\mathfrak{A}}(x) & \text{elsewhere.} \end{cases}$$

Then g is $1_{\mathfrak{A}}$ compatible and preserves the subuniverses of $\mathfrak{A}$. Let t be a term representing g on $\overline{a}$; then $\langle u, v\rangle = \langle g(b), g(c)\rangle = \langle t^{\mathfrak{A}}(b), t^{\mathfrak{A}}(c)\rangle$ is an element of $H(b,c)$.

$(\Leftarrow)$: Let $h : A^n \to A$, $n \geq 1$, be a $1_{\mathfrak{A}}$ compatible function which preserves the subuniverses of $\mathfrak{A}$; let $a \in A$ and $\langle b_i, c_i\rangle \in \overline{a}^2$ for $1 \leq i \leq n$, and consider $B = Sg(b_1, \ldots, b_n)$, $C = Sg(c_1, \ldots, c_n)$, $E = Sg(\langle b_1, c_1\rangle, \ldots, \langle b_n, c_n\rangle)$.

If $b_i = c_i$ for each i, then $E = \triangle_B$, which is preserved by h.

If $b_{i_0} \neq c_{i_0}$ for some i_0, then $E = H(b,c) = (B \times C) \cap 1_{\mathfrak{A}}$ by (ii); since h preserves B, C and $1_{\mathfrak{A}}$, we see that $h^{\mathfrak{A}^2}(\langle b_1, c_1\rangle, \ldots, \langle b_n, c_n\rangle) = \langle h(b), h(c)\rangle$ belongs to $(B \times C) \cap 1_{\mathfrak{A}}$. Since **d** is a Pixley function, we can obtain a majority function which is termal on classes. By Lemma 2.1(i), h is termal on classes. □

Using the function **q** defined in the proof of Lemma 3.1, we see that if $\mathfrak{A}$ is $\star$-semiprimal then **d** is also term representable on classes.

Let $a_1, \ldots, a_m$ be elements of $\bigcup_{u \in A} \overline{u}^n$, and $a^i := \langle a_1(i), \ldots, a_m(i)\rangle$ for $1 \leq i \leq n$, then $K(a^1, \ldots, a^n) := \{x \in A^m; \langle x(k), x(l)\rangle \in H(a_k, a_l) \text{ for } 1 \leq k, l \leq m\}$ is a subuniverse of $\mathfrak{A}^m$, which contains $Sg(a^1, \ldots, a^n)$.

Theorem 3.4 *$\mathfrak{A}$ is $\star$-semiprimal iff the following conditions are satisfied:*
(i) For any non zero natural number n, and distinct elements $b, c \in \bigcup_{u\in A} \overline{u}^n$, if $B = Sg(b(1), \ldots, b(n))$ and $C = Sg(c(1), \ldots, c(n))$, then

$$H(b,c) = \begin{cases} (B \times C) \cap 1_{\mathfrak{A}}, & \text{if } \overline{b(1)} = \overline{c(1)}; \\ (B \times C), & \text{if not.} \end{cases}$$

(ii) For any non zero natural numbers m, n, and elements $a_1, \ldots, a_m$ of $\bigcup_{u \in A} \overline{u}^n$, if $a^i := \langle a_1(i), \ldots, a_m(i)\rangle$ for $1 \leq i \leq n$, then $K(a^1, \ldots, a^n) = Sg(a^1, \ldots, a^n)$ as subuniverses of $\mathfrak{A}^m$.

Proof $(\Rightarrow)$: For (i), we have $H(b,c) \subseteq B \times C$.

If $\overline{b(1)} = \overline{c(1)}$, the result follows from Theorem 3.3.

If $\overline{b(1)} \neq \overline{c(1)}$, let $\langle u, v\rangle \in B \times C$, and t_1, t_2 be n-ary terms such that $u = t_1^{\mathfrak{A}}(b)$ and $v = t_2^{\mathfrak{A}}(c)$; the function $g : A^n \to A$ defined by

$$g(x) = \begin{cases} t_1^{\mathfrak{A}}(x), & \text{if } x \in \overline{b(1)}^n; \\ t_2^{\mathfrak{A}}(x), & \text{elsewhere.} \end{cases}$$

is $1_{\mathfrak{A}}$ compatible and preserves the subuniverses of $\mathfrak{A}$. So there is an n-ary term t representing g on classes; $\langle u, v\rangle = \langle g(b), g(c)\rangle = \langle t^{\mathfrak{A}}(b), t^{\mathfrak{A}}(c)\rangle \in H(b,c)$.

For (ii), we must show that $K(a^1, \ldots, a^n) \subseteq Sg(a^1, \ldots, a^n)$.

Let $y \in K(a^1, \ldots, a^n)$; for each $k \in \underline{m} := \{1, \ldots, m\}$, consider the set $J_k = \{l \in \underline{m}; \overline{a_k(1)} = \overline{a_l(1)}\}$, and let $\sigma(k)$ be the minimum of J_k. If $\alpha, \beta \in J_k$, then $\langle y(\alpha), y(\beta)\rangle \in H(a_\alpha, a_\beta) = Sg(\langle a_\alpha(1), a_\beta(1)\rangle, \ldots, \langle a_\alpha(n), a_\beta(n)\rangle)$.

Since the Pixley function **d** is termal on classes, we can find a majority function which is termal on classes. So from the proof of Lemma 2.1, we can find a term $t_{\sigma(k)}$ such that

$y(l) = t^{\mathfrak{A}}_{\sigma(k)}(a_l)$ for each $l \in J_k$.

Consider the function $h : A^n \to A$ defined by

$$h(x) = \begin{cases} t^{\mathfrak{A}}_{\sigma(k)}(x), & \text{if } x \in \overline{a_k(1)}^n \text{ for some } k \in \underline{m} \ ; \\ x(1), & \text{if } x \notin \overline{a_k(1)}^n \text{ for any } k \in \underline{m} \ . \end{cases}$$

Then h preserves $1_{\mathfrak{A}}$ and the subuniverses of $\mathfrak{A}$; let t be a term representing h on classes; then $y = \langle y(1), \ldots, y(m)\rangle = \langle h(a_1), \ldots, h(a_m)\rangle = t^{\mathfrak{A}^m}(a^1, \ldots, a^n)$, which is an element of $Sg(a^1, \ldots, a^n)$.

($\Leftarrow$): Let $g : A^n \to A$, $n \geq 1$, be such a function. Let $a_1, \ldots, a_m$ be the distinct elements of $\bigcup_{u \in A} \overline{u}^n$, and $a^i := \langle a_1(i), \ldots, a_m(i)\rangle$ for $1 \leq i \leq n$.

For $\underline{b, c} \in \{\underline{a_1, ..., a_m}\}$, let $B = Sg(b(1), ..., b(n))$ and $C = Sg(c(1), ..., c(n))$.

· If $\overline{b(1)} = \overline{c(1)}$, then $\langle g(b), g(c)\rangle \in (B \times C) \cap 1_{\mathfrak{A}}$, and the later is $H(b, c)$.

· If $\overline{b(1)} \neq \overline{c(1)}$, then $\langle g(b), g(c)\rangle \in B \times C = H(b, c)$.

Thus $\langle g(a_1), \ldots, g(a_m)\rangle \in K(a^1, \ldots, a^n) = Sg(a^1, \ldots, a^n)$; so there is an n-ary term t such that $\langle g(a_1), \ldots, g(a_m)\rangle = t^{\mathfrak{A}^m}(a^1, \ldots, a^n)$, and the later is $\langle t^{\mathfrak{A}}(a_1), ..., t^{\mathfrak{A}}(a_m)\rangle$; so t represents g on classes. □

We now examine how *weakly termal functions* can fit with $\star$-semiprimality. We first note that if $\mathfrak{A}$ is $\star$-semiprimal, then every weakly termal function is term representable on classes.

Lemma 3.5 *Every $1_{\mathfrak{A}}$ compatible function $g : A^n \to A, n \geq 1$, which preserves the subuniverses of $\mathfrak{A}$ is weakly termal iff the following conditions are satisfied.*
(i) $\mathfrak{A}$ is weakly $\star$-semiprimal.
(ii) For any distinct elements b, c of A^n such that $\langle b(i), c(i)\rangle \in 1_{\mathfrak{A}}$ for $1 \leq i \leq n$, if $B = Sg(b(1), \ldots, b(n))$ and $C = Sg(c(1), \ldots, c(n))$ then $H(b,c) = (B \times C) \cap 1_{\mathfrak{A}}$.

Proof ($\Rightarrow$): For (i), a weakly termal function is termal on classes; so $\mathfrak{A}$ is weakly $\star$-semiprimal.

For (ii), we already have $H(b, c) = Sg(\langle b(1), c(1)\rangle, \ldots, \langle b(n), c(n)\rangle) \subseteq (B \times C) \cap 1_{\mathfrak{A}}$. Let $\langle u, v\rangle \in (B \times C) \cap 1_{\mathfrak{A}}$, and t_1, t_2 be terms such that $u = t^{\mathfrak{A}}_1(b)$ and $v = t^{\mathfrak{A}}_2(c)$. The function $g : A^n \to A$ defined by

$$g(x) = \begin{cases} t^{\mathfrak{A}}_1(x) & \text{if } x \in (\prod_{1 \leq i \leq n} \overline{b(i)}) \setminus \{c\}; \\ t^{\mathfrak{A}}_2(x) & \text{elsewhere.} \end{cases}$$

is $1_{\mathfrak{A}}$ compatible and preserves the subuniverses of $\mathfrak{A}$; let t be a term representing g on $\prod_{1 \leq i \leq n} \overline{b(i)}$; then $\langle u, v\rangle = \langle t^{\mathfrak{A}}(b), t^{\mathfrak{A}}(c)\rangle$, an element of $H(b, c)$.

($\Leftarrow$): Let $g : A^n \to A$, $n \geq 1$, be such a function. Since $\mathfrak{A}$ is weakly $\star$-semiprimal, there is a majority function which is termal on classes. So by Lemma 2.1.(ii), we only need to prove that g preserves the subuniverses of $1_{\mathfrak{A}}$. Let $\langle b(i), c(i)\rangle \in 1_{\mathfrak{A}}$ for $1 \leq i \leq n$, $B = Sg(b(1), \ldots, b(n))$ and $C = Sg(c(1), \ldots, c(n))$; then $H(b, c) \subseteq (B \times C) \cap 1_{\mathfrak{A}}$, and $\langle g(b), g(c)\rangle \in (B \times C) \cap 1_{\mathfrak{A}}$.

· If $b(i) = c(i)$ for each i, then $\langle g(b), g(c)\rangle \in \triangle_B = H(b, c)$.

· If $b(i) \neq c(i)$ for some i, then $H(b, c) = (B \times C) \cap 1_{\mathfrak{A}}$.

So in any case $\langle g(b), g(c)\rangle \in H(b, c)$. □

If $1_{\mathfrak{A}}$ is hereditarily maximal, condition (ii) of Lemma 3.5 implies that $Con_\star(\mathfrak{B}) = \{\triangle_B, 1_{\mathfrak{B}}\}$ for every substructure $\mathfrak{B}$ of $\mathfrak{A}$. We will refer to this condition as the "$1_{\mathfrak{A}}$ compatible subuniverse property"

We have this immediate corollary from Definition 3.3 and Lemma 3.5.

Corollary 3.6 *Consider the following conditions on $\mathfrak{A}$:*
(i) $\mathfrak{A}$ is $\star$-semiprimal.

(ii) $\mathfrak{A}$ *satisfies the* $1_\mathfrak{A}$ *compatible subuniverse property.*
(iii) Every $1_\mathfrak{A}$ *compatible function* $g : A^n \to A$, $n \geq 1$, *which preserves the subuniverses of* $\mathfrak{A}$ *is weakly termal.*
(iv) Every weakly termal function is term representable on classes.
Then the conjunction of (i) and (ii) is equivalent to the conjunction of (iii) and (iv). □

Below we restate this corollary with the role of the discriminator function highlighted; first let us say that a $1_\mathfrak{A}$ isomorphism $\alpha : \mathfrak{B}/\theta \to \mathfrak{C}/\varphi$ is *full* if $\theta = 1_\mathfrak{A} \cap B^2$. (Note that $\theta = 1_\mathfrak{A} \cap B^2$ iff $\varphi = 1_\mathfrak{A} \cap C^2$)

Theorem 3.7 *Consider the following conditions on* $\mathfrak{A}$.
(i) $\mathfrak{A}$ *is* $\star$*-semiprimal.*
(ii) $\mathfrak{A}$ *satisfies the* $1_\mathfrak{A}$ *compatible subuniverse property.*
(iii) The discriminator function $\mathbf{d}$ *is term representable on classes.*
(iv) Every subuniverse of $1_\mathfrak{A}$ *is an identity or the lifting of a full* $1_\mathfrak{A}$ *isomorphism.*
(v) Every weakly termal function is term representable on classes.
Then the conjunction of (i) and (ii) is equivalent to the conjunction of (iii), (iv) and (v).

Proof $(\Rightarrow)$: (iii) follows from (i) and the proof of Lemma 3.1.(i).

For (iv), let E be a subuniverse of $1_\mathfrak{A}$; then E is the lifting of a $1_\mathfrak{A}$ isomorphism $\alpha : \mathfrak{B}/\theta \to \mathfrak{C}/\varphi$, where $B = \pi_1(E)$ and $C = \pi_2(E)$. Let $\langle b(1), c(1)\rangle, \ldots, \langle b(n), c(n)\rangle$ be the distinct elements of E. If E is not an identity then $b(i) \neq c(i)$ for some i; so $E = H(b, c) = (B \times C) \cap 1_\mathfrak{A}$; thus $\theta = 1_\mathfrak{A} \cap B^2$.

For (v), since $\mathbf{d}$ is termal on classes, it follows from Lemma 2.1.(ii) that a weakly termal function h preserves the subuniverses of $1_\mathfrak{A}$; thus h is $1_\mathfrak{A}$ compatible and preserves the subuniverses of $\mathfrak{A}$.

$(\Leftarrow)$: For (ii), we observe that $H(b, c)$ is a subuniverse of $1_\mathfrak{A}$, and the result follows from (iv).

For (i), let $g : A^n \to A$, $n \geq 1$, be a $1_\mathfrak{A}$ compatible function which preserves the subuniverses of $\mathfrak{A}$, and $\langle b(i), c(i)\rangle \in 1_\mathfrak{A}$ for $1 \leq i \leq n$.

Let $B = Sg(b(1), \ldots, b(n))$ and $C = Sg(c(1), \ldots, c(n))$.

· If $b(i) = c(i)$ for each i, then $\langle g(b), g(c)\rangle \in \triangle_B = H(b, c)$.

· If $b(i) \neq c(i)$ for some i, then $H(b, c)$ is the lifting of a full $1_\mathfrak{A}$ isomorphism $\alpha : \mathfrak{B}/\theta \to \mathfrak{C}/\varphi$; thus $H(b, c) = (B \times C) \cap 1_\mathfrak{A}$, which contains $\langle g(b), g(c)\rangle$; so g is weakly termal. And the result follows from (v). □

In some cases, the $1_\mathfrak{A}$ compatible subuniverse property follows from $\star$-semiprimality, as shown in the proposition below.

Definition 3.4
(i) *A congruence* θ *of* $\mathfrak{A}$ *is called* ***uniform*** *if all* θ *classes have the same cardinality.*
(ii) $\mathfrak{A}$ *is said to be* $\star$***-congruence uniform*** *if every* $\star$*-congruence of* $\mathfrak{A}$ *is uniform.*

Proposition 3.8 *Let* $1_\mathfrak{A}$ *be hereditarily maximal,* $\mathfrak{A}$ *be weakly* $\star$*-semiprimal and hereditarily* $\star$*-congruence uniform. Then* $\mathfrak{A}$ *satisfies the* $1_\mathfrak{A}$ *compatible subuniverse property.*

Proof By Lemma 3.1, $H(b, c) = Sg(\langle b(1), c(1)\rangle, \ldots, \langle b(n), c(n)\rangle)$ is the lifting of a $1_\mathfrak{A}$ semi-identity $\alpha : \mathfrak{B}/\theta \to \mathfrak{C}/\varphi$. Let $i \in \underline{n} = \{1, \ldots, n\}$ such that $b(i) \neq c(i)$; then, $[b(i)]_\theta = \overline{b(i)} \cap B = [b(i)]_{1_\mathfrak{B}}$.
· $\overline{b(i)} \cap B = \{b(i)\}$ implies $[u]_{1_\mathfrak{B}} = \{u\}$ for each $u \in B$, and $1_\mathfrak{B} = \triangle_B = \theta$.
· $\overline{b(i)} \cap B \neq \{b(i)\}$ implies $[b(i)]_\theta \neq \{b(i)\}$; thus $[u]_\theta \neq \{u\}$ for each $u \in B$; since θ is a simple $\star$-congruence, $[u]_\theta = [u]_{1_\mathfrak{B}}$ for each $u \in B$, and $\theta = 1_\mathfrak{B}$.

In any case, we have $\theta = 1_\mathfrak{B} = 1_\mathfrak{A} \cap B^2$; so $\varphi = 1_\mathfrak{A} \cap C^2 = 1_\mathfrak{C}$, and $H(b, c) = (B \times C) \cap 1_\mathfrak{A}$.
□

When $1_{\mathfrak{A}}$ is hereditarily maximal and $\mathfrak{A}$ is hereditarily $\star$-congruence uniform, we obtain in Theorem 3.7 that $\star$-semiprimality is characterized by the conjunction of $(iii), (iv)$ and (v). Condition (iv) may be seen as a generalization of $(SP2)$ in the Introduction.

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