On n-Fold Quasi-Associative Ideals in *BCI-algebras*

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Abstract

We discuss the notion of *n-fold quasi-associative weak ideals* as a natural generalization of *quasi-associative weak ideals*. Then, using the notion of fuzzy point, we give some characterizations of *fuzzy n-fold quasi-associative ideals*. Finally, we construct some algorithms for studying the foldness theory of quasi-associative ideals in *BCI-algebras*.

Key Words: BCI-algebra, Fuzzy point, Fuzzy ideals, Fuzzy n-fold quasi-associative ideals.
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1 Introduction

The class of *BCI-algebras* which was introduced by Iseki and Tanaka [6] is an important class of logical algebras which has two origins. One of the motivations for their use is based on set theory, the other on classical and non-classical propositional calculus. By definition, the notion of *BCI-algebras* generalizes the one of sets with the set subtraction as the only fundamental non nullary operation and the notion of implication algebras. The concept of fuzzy sets introduced by Zadeh [8] was applied to *BCI-algebras* by Xi [7]. Since then, many researchers have investigated various properties of these algebras.

One of the main problems in fuzzy mathematics is how to carry out the ordinary concepts to the fuzzy case. The difficulty lies in how to pick out the rational generalization from the large number of available approaches. It is worth noting that fuzzy ideals are different from ordinary ideals in the sense that one can not say which *BCI-algebra* element either belongs or not to the fuzzy ideal under consideration.

In this paper, we discuss the notion of *n-fold quasi-associative weak ideals* as a natural generalization of *quasi-associative weak ideals*. Using the *fuzzy point* notion, we give some characterizations of *fuzzy n-fold quasi-associative ideals*. First, given any set X, we construct a set of fuzzy points $\tilde{X}$. Then, for any fuzzy subset A of X, we construct a subset $\tilde{A}$ of $\tilde{X}$ and establish some similarities between A and $\tilde{A}$.

This approach of fuzzy *BCI-algebras*, which is based on the concept of fuzzy point has many advantages. It provides a convenient method which may be

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useful in attacking various problems in fuzzy algebra and other areas of fuzzy mathematics. It enables us to develop adequate theory of fuzzy *BCI-algebras* parallel to the ordinary *BCI-algebra* theory. It provides also some links between the classical fuzzy approach and the ordinary case. Through these similarities, we can reformulate many concepts and results of the ordinary *BCI-algebras* theory and carry them to fuzzy case in a natural way.

2 Preliminaries

A *BCI-algebra* is a nonempty set X with a binary operation $*$ and a constant 0 satisfying the following axioms:

BCI-1. $[(x * y) * (x * z)] * (z * y) = 0;$

BCI-2. $[x * (x * y)] * y = 0;$

BCI-3. $x * x = 0;$

BCI-4. $x * y = 0$ *and* $y * x = 0 \Longrightarrow \; x = y.$

A partial ordering $\leq$ can be defined on X by

BCI-5. $x \leq y \Longleftrightarrow x * y = 0.$

Therefore, $(X, \leq)$ is a partially ordered set with the least element 0. The following properties also hold in any *BCI-algebra* ([1],[2], [3],[4], [5]).

1) $x * 0 = x;$

2) $0 * (x * y) = (0 * x) * (0 * y);$

3) $x * y = 0 \Longrightarrow (x * z) * (y * z) = 0$ and $(z * y) * (z * x) = 0;$

4) $(x * y) * z = (x * z) * y;$

5) $(x * y) * x = 0;$

6) $x * (x * (x * y)) = x * y.$

We briefly review some fuzzy logic concepts and refer the reader to [2], for more detail.

Definition 2.1. *A fuzzy subset of a BCI-algebra X is a function*

$$\mu : X \longrightarrow [0, 1].$$

If ξ is the family of all fuzzy sets on X, $x_\lambda \in \xi$ is a fuzzy point iff $\forall y \in X$,

$$x_\lambda(y) = \begin{cases} \lambda & \textit{if} \;\; x = y, \\ 0 & \textit{otherwise}. \end{cases}$$

We denote by $\tilde{X} = \{x_\lambda : x \in X, \lambda \in (0,1]\}$ the set of all fuzzy points on X and we define a binary operation on $\tilde{X}$ as follows:

$$x_\lambda * y_\mu = (x * y)_{\min(\lambda,\mu)}. \quad (1)$$

It is easy to verify that for any x_λ, y_μ, $z_\alpha \in \tilde{X}$, the following conditions hold:

BCI-1'. $[(x_\lambda * y_\mu) * (x_\lambda * z_\alpha)] * (z_\alpha * y_\mu) = 0_{\min(\lambda,\mu,\alpha)}$;

BCI-2'. $[x_\lambda * (x_\lambda * y_\mu)] * y_\mu = 0_{\min(\lambda,\mu)}$;

BCI-3'. $x_\lambda * x_\mu = 0_{\min(\lambda,\mu)}$.

Remark 2.1. *The condition* **BCI-4.** *is not true in* $(\tilde{X}, *)$. *So the partial order* $\leq$ *on* $(X, *)$ *can not be extended to* $(\tilde{X}, *)$.

$\forall\ x_\lambda$, y_μ, $z_\alpha \in \tilde{X}$, we can also establish the following conditions:

1') $x_\lambda * 0_\mu = x_{\min(\lambda,\mu)}$;

2') $0_\alpha * (x_\lambda * y_\mu) = (0_\alpha * x_\lambda) * (0_\alpha * y_\mu)$;

3')

$$x_\lambda * y_\mu = 0_{\min(\lambda,\mu)} \Longrightarrow \begin{cases} (x_\lambda * z_\alpha) * (y_\mu * z_\alpha) = 0_{\min(\lambda,\mu,\alpha)} \text{ and} \\ (z_\alpha * y_\mu) * (z_\alpha * x_\lambda) = 0_{\min(\lambda,\mu,\alpha)}; \end{cases}$$

4') $(x_\lambda * y_\mu) * z_\alpha = (x_\lambda * z_\alpha) * y_\mu$;

5') $(x_\lambda * y_\mu) * x_\lambda = 0_{\min(\lambda,\mu)}$;

6') $x_\lambda * (x_\lambda * (x_\lambda * y_\mu)) = x_\lambda * y_\mu$.

We recall that if A is a fuzzy subset of a *BCI-algebra* X, then we have the following:

$$\tilde{A} = \{x_\lambda \in \tilde{X} : A(x) \geq \lambda, \lambda \in (0,1]\}.$$

We denote $(...((x * y) * y) * ...) * y$ by $x * y^n$ and $(...((x_\lambda * y_\mu) * y_\mu) * ...) * y_\mu$ by $x_\lambda * y_\mu^n$ (where respectively y and y_μ occurs n times) with $x, y \in X$ and $x_\lambda, y_\mu \in \tilde{X}$.

Definition 2.2. *An ideal of a BCI-algebra X is a subset I containing 0 such that:*

$$x * y \in I \text{ and } y \in I \Longrightarrow x \in I. \quad (2)$$

An ideal I is said to be closed if $0 * x \in I$ when $x \in I$.

Definition 2.3. *A fuzzy subset A of X is a fuzzy ideal if it satisfies*

$$A(0) \geq A(x) \text{ and } A(x) \geq \min(A(x * y), A(y))\ \forall\ x, y, z \in X. \quad (3)$$

A fuzzy ideal A is closed if $A(0 * x) \geq A(x)$ for any $x \in X$.

Definition 2.4. *Let A be a fuzzy subset of X and $\tilde{A}$ a subset of $\tilde{X}$. If for any $\theta \in Im(A)$, $0_\theta \in \tilde{A}$ and for any $x_\lambda, y_\mu \in \tilde{X}$:*

$$x_\lambda * y_\mu \in \tilde{A} \text{ and } y_\mu \in \tilde{A} \Longrightarrow x_{\min(\lambda,\mu)} \in \tilde{A}, \tag{4}$$

then $\tilde{A}$ is called a weak ideal of $\tilde{X}$.
A weak ideal $\tilde{A}$ is closed if for any $\alpha \in Im(A)$ $0_\alpha * x_\lambda \in \tilde{A}$ when $x_\lambda \in \tilde{A}$.

The following theorem gives some characterizations of fuzzy ideals.

Theorem 2.1. *[2] Suppose that A is a fuzzy subset of a BCI-algebra X, then the following conditions are equivalent:*

1. *A is a fuzzy ideal;*
2. *$\forall\ x_\lambda, y_\mu \in \tilde{A}, (z_\alpha * y_\mu) * x_\lambda = 0_{\min(\lambda,\mu,\alpha)} \Longrightarrow z_{\min(\lambda,\mu,\alpha)} \in \tilde{A}$;*
3. *$\forall\ t \in (0,1]$, the t-level subset $A_t = \{x \in X : A(x) \geq t\}$ is an ideal of X when $A_t \neq \emptyset$;*
4. *$\tilde{A}$ is a weak ideal of $\tilde{X}$.*

3 Fuzzy n-fold quasi-associative weak ideals

Definition 3.1. *Let I be a subset of a BCI-algebra X. If $0 \in I$ and $\exists\ n \in \mathbb{N}$ such that*

$$x * (y * z^n) \in I \text{ and } y \in I \Longrightarrow x * z^n \in I, \tag{5}$$

then I is said to be an n-fold quasi-associative ideal.

Definition 3.2. *A fuzzy subset A of X is a fuzzy n-fold quasi-associative ideal if it satisfies $A(0) \geq A(x)$ and $\exists\ n \in \mathbb{N}$ such that*

$$A(x * z^n) \geq \min(A(x * (y * z^n)), A(y))\ \forall\ x, y, z \in X. \tag{6}$$

If $\forall\ \theta \in Im(A)$, $0_\theta \in \tilde{A}$ and $\forall\ x_\lambda, y_\mu, z_\alpha \in \tilde{X}$, $\exists\ n \in \mathbb{N}$ such that

$$x_\lambda * (y_\mu * z_\alpha^n) \in \tilde{A} \text{ and } y_\mu \in \tilde{A} \Longrightarrow (x * z^n)_{\min(\lambda,\mu,\alpha)} \in \tilde{A}, \tag{7}$$

then $\tilde{A}$ is an n-fold quasi-associative weak ideal of $\tilde{X}$.

Example 3.1. *Let $X = \{0, 1, 2, 3\}$ with $*$ defined by the following table:*

*	0	1	2	3
0	0	3	0	1
1	1	0	1	3
2	2	3	0	1
3	3	1	3	0

It is easy to check that X is a BCI-algebra. Let $t_1, t_2 \in (0,1]$ and A a fuzzy subset on X defined by $t_1 = A(0) = A(1) = A(3) > A(2) = t_2$.

$$\tilde{A} = \{0_\lambda, \lambda \in (0,t_1]\} \cup \{1_\lambda, \lambda \in (0,t_1]\} \cup \{3_\lambda, \lambda \in (0,t_1]\} \cup \{2_\lambda, \lambda \in (0,t_2]\}.$$

By simple computations, one can realize that $\tilde{A}$ is an 1-fold quasi-associative weak ideal.

Example 3.2. *Consider* $X = \{0, 1, 2, 3\}$ *with* $*$ *defined as in Example 3.1. Let* $t \in [0, 1)$ *and* A *a fuzzy subset on* X *defined by* $1 = A(0)$ *and* $A(1) = A(2) = A(3) = t$,

$$\tilde{A} = \{0_\lambda, \lambda \in (0, 1]\} \cup \{1_\lambda, \lambda \in (0, t]\} \cup \{2_\lambda, \lambda \in (0, t]\} \cup \{3_\lambda, \lambda \in (0, t]\}.$$

$\tilde{A}$ *is not an 1-fold quasi-associative weak ideal because* $0_1 = 1_1 * (0_1 * 3_1) \in \tilde{A}$ *and* $0_1 \in \tilde{A}$*, but* $(1 * 3)_1 = 3_1 \notin \tilde{A}$.

Example 3.3. *Let* $X = \{0, 1, 2, 3\}$ *with* $*$ *defined by the following table:*

$*$	0	1	2	3
0	0	0	2	2
1	1	0	3	2
2	2	2	0	0
3	3	2	1	0

One can easily check that X *is a BCI-algebra. Let* $t_1, t_2 \in (0, 1]$ *and* A *a fuzzy subset on* X *defined by* $t_1 = A(0) = A(1) > A(2) = A(3) = t_2$.

$$\tilde{A} = \{0_\lambda, \lambda \in (0, t_1]\} \cup \{1_\lambda, \lambda \in (0, t_1]\} \cup \{2_\lambda, \lambda \in (0, t_2]\} \cup \{3_\lambda, \lambda \in (0, t_2]\}.$$

By simple calculations one can prove that $\tilde{A}$ *is an 2-fold quasi-associative weak ideal.*

Example 3.4. *Let* $X = \{0, 1, 2\}$ *with* $*$ *defined by the following table:*

$*$	0	1	2
0	0	0	2
1	1	0	2
2	2	2	0

It is easy to check that X *is a BCI-algebra. Let* $t_1, t_2 \in (0, 1]$ *and* A *a fuzzy subset on* X *defined by* $t_1 = A(0) = A(1) > A(2) = t_2$*, simple calculations prove that* $\tilde{A}$ *is an* n*-fold quasi-associative weak ideal for any* $n \geq 0$.

Theorem 3.1. *A fuzzy subset* A *of a BCI-algebra* X *is a fuzzy* n*-fold quasi-associative ideal iff* $\tilde{A}$ *is an* n*-fold quasi-associative weak ideal.*

Proof. Suppose that A is a fuzzy n-fold quasi-associative ideal.
Let $\lambda \in Im(A)$ and suppose that $\lambda = A(x)$. Since A is a fuzzy n-fold quasi-associative ideal, we have $A(0) \geq A(x) = \lambda$, hence $0_\lambda \in \tilde{A}$.
Let $x_\lambda * (y_\mu * z_\alpha^n) \in \tilde{A}$ and $y_\mu \in \tilde{A}$ then,

$$A(x * (y * z^n)) \geq \min(\lambda, \mu, \alpha) \text{ and } A(y) \geq \mu.$$

Since A is a fuzzy n-fold quasi-associative ideal, we have

$$A(x * z^n) \geq \min(A(x * (y * z^n)), A(y)) \geq \min(\lambda, \mu, \alpha).$$

Hence, $(x * z^n)_{\min(\lambda,\mu,\alpha)} \in \tilde{A}$.

Conversely, let $x \in X$ and $\lambda = A(x)$, $\lambda \in Im(A)$. Since $\tilde{A}$ is an n-fold quasi-associative weak ideal, we have $0_\lambda \in \tilde{A}$. Therefore, $A(0) \geq \lambda = A(x)$.
Let $x, y, z \in X$, if $A(x * (y * z^n)) = \beta$ and $A(y) = \alpha$ then,

$$(x * (y * z^n))_{\min(\beta,\alpha)} = x_\beta * (y_\alpha * z_\beta^n) \in \tilde{A} \text{ and } y_\alpha \in \tilde{A}.$$

Since $\tilde{A}$ is n-fold quasi-associative weak ideal, we have $(x * z^n)_{\min(\beta,\alpha)} \in \tilde{A}$. So,

$$A(x * z^n) \geq \min(\beta, \alpha) = \min(A(x * (y * z^n)), A(y)).$$

□

We describe below the relation between weak ideals and n-fold quasi-associative weak ideals.

Proposition 3.1. *If $\tilde{A}$ is an n-fold quasi-associative weak ideal, then $\tilde{A}$ is a weak ideal.*

Proof. It is clear that for any $\lambda \in Im(A)$, $0_\lambda \in \tilde{A}$.
Let $x_\lambda * y_\mu \in \tilde{A}$ and $y_\mu \in \tilde{A}$. Since $x_\lambda * y_\mu = x_\lambda * (y_\mu * 0^n_\mu) \in \tilde{A}$, by using the definition of n-fold quasi-associative weak ideal we obtain that $\tilde{A}$ is a weak ideal. □

Corollary 3.1. *If A is a fuzzy n-fold quasi-associative ideal, then A is a fuzzy ideal.*

Proposition 3.2. *If $\tilde{A}$ is an n-fold quasi-associative weak ideal, then $\tilde{A}$ is a closed weak ideal.*

Proof. Let $x_\lambda \in \tilde{A}$ and $\alpha \in Im(A)$. We have

$$(0_\alpha * x^n_\lambda) * (0_\alpha * x^n_\lambda) = 0_{\min(\lambda,\alpha)} \in \tilde{A} \text{ and } 0_\alpha \in \tilde{A}.$$

Since $\tilde{A}$ is an n-fold quasi-associative weak ideal, we have also $(0_\alpha * x^n_\lambda) * x^n_\lambda \in \tilde{A}$. By the fact that $\tilde{A}$ is a weak ideal $2n - 1$ times, it follows that $\tilde{A}$ is a closed weak ideal. □

Corollary 3.2. *If A is a fuzzy n-fold quasi-associative ideal, then A is a closed fuzzy ideal.*

Now, we can characterize fuzzy n-fold quasi-associative ideal in terms of level subsets as follows:

Theorem 3.2. *A fuzzy subset A of a BCI-algebra X is a fuzzy n-fold quasi-associative ideal iff $A^t = \{x \in X/A(x) \geq t\}$ is either empty or an n-fold quasi-associative ideal of X.*

Proof. Assume that A is a fuzzy n-fold quasi-associative ideal of X.
Let $t \in (0, 1]$ and $x \in A^t$, $A(x) \geq t$. Since A is a fuzzy ideal, $A(0) \geq A(x)$, hence $0 \in A^t$.
Let $x, y, z \in X$ with $x * (y * z^n) \in A^t$ and $y \in A^t$, then

$$A(x * (y * z^n)) \geq t \text{ and } A(y) \geq t.$$

Since A is a fuzzy n-fold quasi-associative ideal, we have

$$A(x * z^n) \geq \min(A(x * (y * z^n)), A(y)) \geq t.$$

Hence $x * z^n \in A^t$.
This proves that the t-level set A^t is an n-fold quasi-associative ideal of X.

Conversely, assume that $A^t = \{x \in X/A(x) \geq t\}$ is an n-fold quasi-assocative ideal of X.
We want to prove that A is a fuzzy n-fold quasi-associative ideal:
It is easy to prove that $\forall\ x \in X$, $A(0) \geq A(x)$.
We need to show that

$$A(x * z^n) \geq \min(A(x * (y * z^n)), A(y))\ \forall\ x, y, z \in X. \tag{8}$$

If the inequality (8) is not satisfied, then $\exists\ a, b, c \in X$ such that

$$A(a * c^n) < \min(A(a * (b * c^n)), A(b)).$$

When we set

$$t_0 = \frac{1}{2}[A(a * c^n) + \min(A(a * (b * c^n)), A(b))],$$

we have

$$A(a * c^n) < t_0 < \min(A(a * (b * c^n)), A(b)).$$

We obtain $a*c^n \notin A^{t_0}$, but $a*(b*c^n) \in A^{t_0}$ and $b \in A^{t_0}$. That is a contradiction since A^{t_0} is an n-fold quasi-associative ideal of X. Therefore A is a fuzzy n-fold quasi-associative ideal and the proof is complete. □

Corollary 3.3. *If A is a fuzzy n-fold quasi-associative ideal of X, then $X_A = \{x \in X : A(x) = A(0)\}$ is an n-fold quasi-associative ideal of X.*

The following theorem gives a characterization of n-fold quasi-associative weak ideals.

Theorem 3.3. *If $\tilde{A}$ is a weak ideal (namely A is a fuzzy ideal by Theorem 2.1), then the following conditions are equivalent:*

1) *A is a fuzzy n-fold quasi-associative ideal.*
2) *$\forall\ x_\lambda, y_\mu \in \tilde{X}$ and $\alpha \in Im(A)$, $x_\lambda * (0_\alpha * y_\mu^n) \in \tilde{A} \Longrightarrow x_{\min(\lambda,\alpha)} * y_\mu^n \in \tilde{A}$.*
3) *$A(x * y^n) \geq A(x * (0 * y^n))$.*
4) *$x_\lambda * (y_\mu * z_\alpha^n) \in \tilde{A} \Longrightarrow (x_\lambda * y_\mu) * z_\alpha^n \in \tilde{A}$.*
5) *$A((x * y) * z^n) \geq A(x * (y * z^n))$.*
6) *$\tilde{A}$ is an n-fold quasi-associative weak ideal of $\tilde{X}$.*

Proof.

1) ⇒ 2) Let $x_\lambda * (0_\alpha * y_\mu^n) \in \tilde{A}$, we have $A(x * (0 * y^n)) \geq \min(\lambda, \mu, \alpha)$. Since A is a fuzzy n-fold quasi-associative ideal, we obtain

$$A(x * y^n) \geq A(x * (0 * y^n)) \geq \min(\lambda, \mu, \alpha).$$

Therefore, $x_{\min(\lambda,\alpha)} * y_\mu^n = (x * y^n)_{\min(\lambda,\mu,\alpha)} \in \tilde{A}$.

2) ⇒ 3) Let $x, y \in X$ and $\theta = A(x * (0 * y^n))$, then

$$x_\theta * (0_\theta * y_\theta^n) = (x * (0 * y^n))_\theta \in \tilde{A}.$$

Using the hypothesis, we obtain $(x * y^n)_\theta = x_\theta * y_\theta^n \in \tilde{A}$. So,

$$A(x * y^n) \geq \theta = A(x * (0 * y^n)).$$

3) $\Rightarrow$ 4) Let $x_\lambda * (y_\mu * z_\alpha^n) \in \tilde{A}$. We have $A(x * (y * z^n)) \geq \min(\lambda, \mu, \alpha)$. We also observe that $(x * y) * (0 * z^n) \leq x * (y * z^n)$. By using the fact that A is a fuzzy ideal we obtain

$$A((x * y) * (0 * z^n) \geq A(x * (y * z^n)).$$

Combining this result with the hypothesis, we obtain

$$A((x * y) * z^n) \geq A((x * y) * (0 * z^n)) \geq A(x * (y * z^n)) \geq \min(\lambda, \mu, \alpha).$$

Thus $(x_\lambda * y_\mu) * z_\alpha^n \in \tilde{A}$.

4) $\Rightarrow$ 5) Let $x, y, z \in X$ and $\theta = A(x * (y * z^n))$. Then $x_\theta * (y_\theta * z_\theta^n) \in \tilde{A}$. By the hypothesis, we have $(x_\theta * y_\theta) * z_\theta^n \in \tilde{A}$. Hence

$$A((x * y) * z^n) \geq \theta = A(x * (y * z^n))$$

.

5) $\Rightarrow$ 6) Let $x_\lambda * (y_\mu * z_\alpha^n) \in \tilde{A}$ and $y_\mu \in \tilde{A}$. We have

$$A(x * (y * z^n)) \geq \min(\lambda, \mu, \alpha) \text{ and } A(y) \geq \mu.$$

Combining the fact that A is a fuzzy ideal and the hypothesis, we obtain

$$A(x * z^n) \geq \min(A((x * z^n) * y), A(y)) = \min(A((x * y) * z^n), A(y))$$

and $\min(A((x*y)*z^n), A(y)) \geq \min(A(x*(y*z^n)), A(y)) \geq \min(\lambda, \mu, \alpha)$. So, $(x * z^n)_{\min(\lambda,\mu,\alpha)} \in \tilde{A}$.

6) $\Rightarrow$ 1) Follows from Theorem 3.1.

□

Theorem 3.4. *If $\tilde{A}$ is a weak ideal (namely A is a fuzzy ideal by Theorem 2.1) and k is a positive integer then the following conditions are equivalent:*

1. *A is a fuzzy n-fold quasi-associative ideal.*
2. *$\forall\ x_\lambda, y_\mu, z_\alpha \in \tilde{X}$ and $\theta \in Im(A)$, $(x_\lambda * z_\alpha^k) * (0_\theta * y_\mu^n) \in \tilde{A} \Longrightarrow (x_\lambda * z_\alpha^k) * y_{\min(\theta,\mu)}^n \in \tilde{A}$.*
3. *$A((x * z^k) * y^n) \geq A((x * z^k) * (0 * y^n))$.*
4. *$(x_\lambda * z_\alpha^k) * (0_\theta * y_\mu^n) \in \tilde{A}$ and $z_\alpha \in \tilde{A} \Longrightarrow x_\lambda * y_{\min(\mu,\alpha,\theta)}^n \in \tilde{A}$.*
5. *$A(x * y^n) \geq \min(A((x * z^k) * (0 * y^n)), A(z))$.*
6. *$\tilde{A}$ is an n-fold quasi-associative weak ideal of $\tilde{X}$.*

The proof is similar to that of the Theorem 3.3 and is omitted.

Lemma 3.1. *[3] (Extension theorem of n-fold quasi-associative ideals) If I and J are two ideals such that $I \subseteq J$, and I is n-fold quasi-associative, then J is also n-fold quasi-associative.*

Theorem 3.5. *(Extension theorem of fuzzy n-fold quasi-associative ideals) If A and B are two fuzzy ideals such that $A \subseteq B$, $A(0) = B(0)$ and A is fuzzy n-fold quasi-associative, then B is also fuzzy n-fold quasi-associative*

Proof. To prove that B is fuzzy n-fold quasi-associative, it suffices to show that $\forall\ t \in (0,1]$, the t-level subset $B^t = \{x \in X : B(x) \geq t\}$ is an n-fold quasi-associative ideal of X when $B^t \neq \emptyset$.
Since $\forall\, t \in (0,1]$, $A^t \subseteq B^t$, we apply the Theorem 3.2 and the extension theorem of n-fold quasi-associative ideals (Lemma 3.1) and obtain the result. □

Corollary 3.4. *(Extension theorem of n-fold quasi-associative weak ideals) If $\tilde{A}$ and $\tilde{B}$ are two weak ideals such that $\tilde{A} \subseteq \tilde{B}$, $A(0) = B(0)$ and $\tilde{A}$ is n-fold quasi-associative, then $\tilde{B}$ is also n-fold quasi-associative.*

4 Conclusion and further suggestions

We have characterized the notion of quasi-associative ideals and we give in appendix some algorithms for studying n-fold quasi-associative ideals and their fuzzifications. These ideas may also be used to study the foldness of other deductive systems in BCI/BCK/BL/MV algebras.

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References

[1] J. Meng, Y. B. Jun and H. S. Kim, *Fuzzy Implicative Ideals in BCK-algebras*, Fuzzy Sets and Systems, **86** (1997), pp. 243-248.

[2] C. Lele, C. Wu, P. Weke, T. Mamadou and G. E. Njock, *Fuzzy Ideals and Weak Ideals in BCK-algebra*, Math. Japon, **4** (2001), pp. 599-612.

[3] Y. B. Jun, S. Z. Song and C. Lele, *Foldness of Quasi-associative Ideals in BCI-algebras*, Scientiae Mathematicae, **6** (2002), pp. 227-231.

[4] Y. B. Jun and K. H. Kim, *Fuzzifications of Foldness of Quasi-associative Ideals in BCI-algebras*, J. Appl. Math. Computing, **11** (2003), pp. 255-263.

[5] Y. Huang and Z. Chen, *On Ideals in BCK-algebra*, Math. Japonica, **50** (1999), pp. 211-226.

[6] K. Iseki and S. Tanaka, *An Introduction to the Theory of BCK-algebras*, Math. Japonica, **23** (1978), pp. 935-942.

[7] O. G. Xi, *Fuzzy BCK-algebras*, Math. Japonica, **36** (1991), pp. 935-942.

[8] L. A. Zadeh, *Fuzzy Sets*, Inform. and Control, **8** (1965), pp. 338-353.

A Algorithms

Algorithm for *BCI-algebras*

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Input(X : set, ∗: binary operation)
Output("X is a BCI-algebra or not")
Begin
  If X = ∅ then
    go to (1.);
  EndIf
  If 0 ∉ X then
    go to (1.);
  EndIf
  Stop:=false;
  i := 1;
  While i ≤ |X| and not(Stop) do
    If x_i ∗ x_i ≠ 0 then
      Stop:=true;
    EndIf
    j := 1
    While j ≤ |X| and not(Stop) do
      If x_i ∗ (x_i ∗ y_j) ≠ 0 then
        Stop:=true;
      EndIf
      If (x_i ∗ y_j = 0) and (y_j ∗ x_i = 0) then
        If x_i ≠ y_j then
          Stop:=true;
        EndIf
      EndIf
      k := 1;
      While k ≤ |X| and not(Stop) do
        If ((x_i ∗ y_j) ∗ (x_i ∗ z_k)) ∗ (z_k ∗ y_j) ≠ 0 then
          Stop:=true;
        EndIf
      EndWhile
    EndWhile
  EndWhile
  If Stop then
    (1.) Output("X is not a BCI-algebra")
  Else
    Output("X is a BCI-algebra")
  EndIf
End
```

Algorithm for n-fold quasi-associative ideals

```
Input(X : BCI-algebra, * : binary operation, I ∈ X, n ∈ ℕ);
Output("I is an n-fold quasi-associative ideal of X or not");
Begin
  If 0 ∉ I then
     go to (1.);
  EndIf
  Stop:=false;
  i := 1;
  While i ≤ |X| and not(Stop) do
     j := 1
    While j ≤ |X| and not(Stop) do
      k := 1;
      While k ≤ |X| and not(Stop) do
        If (x_i * (y_j * z_k^n) ∈ I) and (y_j ∈ I)then
          If (x_i * z_k^n) ∉ I then
            Stop:=true;
          EndIf
        EndIf
      EndWhile
    EndWhile
  EndWhile
  If Stop then
    Output("I is not an n-fold quasi-associative ideal of X")
  Else
    Output("I is an n-fold quasi-associative ideal of X")
  EndIf
End
```

Algorithm for fuzzy subsets

```
Input(X : BCI-algebra, A : X ⟶ [0, 1]);
Output("A is a fuzzy subset of X or not");
Begin
  Stop:=false;
  i := 1;
  While i ≤ |X| and not(Stop) do
    If (A(x_i) < 0) or (A(x_i) > 1) then
      Stop:=true;
    EndIf
  EndWhile
  If Stop then
    Output("A is a fuzzy subset of X")
  Else
    Output("A is not a fuzzy subset of X")
  EndIf
End
```

Algorithm for fuzzy n-fold quasi-associative ideals

```
Input(X : BCI-algebra, ∗ : binary operation, A : fuzzy subset of X, n ∈ ℕ);
Output("A is a fuzzy n-fold quasi-associative ideal of X or not");
Begin
 Stop:=false;
 i := 1;
 While i ≤ |X| and not(Stop) do
  If A(0) < A(x_i) then
   Stop:=true;
  EndIf
   j := 1
  While j ≤ |X| and not(Stop) do
   k := 1;
   While k ≤ |X| and not(Stop) do
    If A(x_i ∗ z_k^n) < Min(A(x_i ∗ (y_j ∗ z_k^n)), A(y_j)) then
     Stop:=true;
    EndIf
   EndWhile
  EndWhile
 EndWhile
 If Stop then
  Output("A is not a fuzzy n-fold quasi-associative ideal of X")
 Else
  Output("A is a fuzzy n-fold quasi-associative ideal of X")
 EndIf
End
```