



## On the Minimum Norm Solution to Weber Problem

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### Abstract

This paper investigates an unconstrained form of classical Weber problem. The main idea is to reformulate Weber problem as unconstrained minimum norm problem. A result based on the representation of the objective function as a Lipschitzian function, which is necessarily a convex function, is proposed. The existence of global solution to such problem is proven using coercivity assumptions.

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## 1 Introduction

The continuous single-facility location problem can be stated as follows: find the location for a new facility  $X = (x_f, y_f)$  such that the sum of the weighted distances from  $X$  to  $n$  existing facility locations ( $P_i = (x_i, y_i), i = 1, \dots, n$ ) is minimized.

If we let  $d(X, P_i)$  represent the distance between  $X$  and  $P_i$  and let  $w_i$  represent the positive weight associated with travel between  $P_i$  and  $X$ , then the problem can be formulated as:

$$\min_X f(X) = \sum_{i=1}^n w_i d(X, P_i) \quad (1)$$

When distance is measured using the Euclidean-distance metric, i.e.

$$d(X, P_i) = ((x_f - x_i)^2 + (y_f - y_i)^2)^{1/2} \quad (2)$$

This problem is sometimes called the Weber problem. The Weber problem (WP) has two very important properties. First,  $f(X)$  is a convex function which ensures that any local optimum is also a global optimum. Second, the optimal location for the new facility must lie within the convex hull of the existing facility locations.

In recent times, research have been carried out on the minimum norm problems and resolved (solved) using several techniques (see [1,3,4,5,9]) to mention a few. Such problems has been found useful in approximation theory, statistical estimation problem [9], signal and image reconstruction as well as in other engineering applications [3].

In the work of [3], the author convinced that minimum norm problem can be recast as fixed point problem and shown the existence and uniqueness of the minimum solution of operator equation of the form  $x = Tx$ , if  $T$  is non expansive. The research carried out by [4] was to find the minimum norm solution of a linear programs by a Newton-type method which was shown to be globally convergent. In [1], the equivalency of this type of problem was shown using duality principle.

Recently, the work of [9] was directed at an estimation problem using simple random sampling technique. The idea was formulation of an estimation problem as an equivalent minimum norm problem in Hilbert space and resolved by an appropriate application of the classical projection theorem.

In this paper, we show that a facility location problem otherwise called Weber problem can be recast as a minimum norm problem and resolve for a global solution using coercivity assumptions.

## 2 Unconstrained Minimization Reformulation

In order to find an unconstrained minimization reformulation for the minimum norm solution for Weber problem (1), we exploit the result of Boyd and Vanderberghe [2] as follows:

As an example application, we can think of the points as locations of plants or warehouses of a company, and the links as the routes over which goods must be shipped. The goal is to find locations that minimize the total transportation cost. In another application, the points represent wires that connect pairs of cell. Here, the goal might be to place the cells in such a way that the total length of wire used to interconnect the cells is minimized.

In the simplest version of the problem the cost associated with arc  $(i, j)$  is the distance between nodes  $i$  and  $j$ :  $f_{ij}(x_i, x_j) = \|x_i - x_j\|$ , i.e. we minimize

$$\sum_{(i,j) \in A} \|x_i - x_j\| \quad (3)$$

We can use any norm, but the most common applications involve the Euclidean or the  $l_1$ -norm. We can include nonnegative weights that reflect differences in the cost per unit distance along different arcs:

$$\sum_{(i,j) \in A} \|x_i - x_j\| \quad (4)$$

By assigning a weight  $w_{ij} = 0$  to pairs of nodes that are not connected and  $w_{ij} = 1$ , otherwise, we can express this problem more simply using the objective

$$\sum_{i < j} w_{ij} \|x_i - x_j\| \quad (5)$$

Thus the Weber problem Eqn.(1) can be formulated as:

$$\min_X f(X) = \min \sum_{i=1}^n w_i d(X, P_i) = \min \|X - P_i\| \quad (6)$$

Let  $D \subset (X, \| \cdot \|)$  be a non-empty set and consider the distance function

$$d_D : X \rightarrow \mathfrak{R}, \quad d_D(u) = d_D(u, c) = \inf \{ \| u - c \| : c \in D \} \quad (7)$$

Having  $u \in X$ , an important problem consists in determining the set

$$P_D(u) = \{ u \in D \mid \bar{u} - u \| = d_D(u) \} \quad (8)$$

$\bar{u} \in P_D(x)$  is called a minimum solution of  $x$  by elements of  $D$ .

**PROPOSITION 2.1**

Let  $d_D : X \rightarrow \mathfrak{R}$  be a distance function and  $D \subset (X, \| \cdot \|)$  be a non-empty set. Then  $d_D$  is Lipschitzian.

Proof of proposition 2.1

$$\text{Let } d_D(u) = \inf \{ \| u - c \| : c \in D \}, \quad d_D(y) = \inf \{ \| y - c \| : c \in D \} \quad (9)$$

$$d_D(u) - d_D(y) = \| u - c \| - \| y - c \| \leq \| u - y \| \quad (10)$$

Using triangle inequality

$$| d_D(u) - d_D(y) | = \| u - c \| - \| y - c \| \leq \| u - y \| \quad (11)$$

$$\Rightarrow | d_D(u) - d_D(y) | \leq \| u - y \| \quad (12)$$

for all  $u, y \in X$

$\Rightarrow d_D$  is Lipschitzian with Lipschitz constant 1.

### 3 Main Existence and Uniqueness Theorems

We shall denote such a problem by

(P)

$$\min f(x), x \in C \quad (13)$$

To problem (P), we can associate a problem (apparently) without constraints

( $\bar{P}$ )

$$\min \bar{f}(x), x \in X \quad (14)$$

where  $\bar{f} = f + l_C$ ,

we call value of problem (P) the extended real

$$v(P) = v(f, C) = \inf\{f(x) \mid x \in C\} \in \mathfrak{R} \quad (15)$$

we call optimal solution of problem (P) an element  $\bar{x} \in C$  with the property that  $f(\bar{x}) = v(P)$ . We denote by  $S(P)$  or  $S(f, C)$  the set of optimal solutions of problem (P). Therefore

$$S(P) = \{\bar{x} \in C \mid \forall x \in X; f(\bar{x}) \leq f(x)\} = \{\bar{x} \in X \mid \forall x \in X; \bar{f}(x) \leq f(x)\} = S(\bar{P}) \quad (16)$$

If  $C \cap \text{dom}f \neq \emptyset$ . The set  $S(f, C)$  is denoted by  $\text{argmin}f$ .

The most important result which assures the existence of minimum solution for (1) is the famous Weierstrass's theorem. But we may use for the same purposes, some coercivity conditions because the underlying spaces are not compact. It is obvious that  $f$  is coercive if and only if all level sets  $[f \leq \lambda]$  are bounded; when  $f$  is convex then  $f$  is coercive if and only if the level set  $[f \leq \lambda]$  is bounded for some  $\lambda > \inf f$ .

### DEFINITION 3.1

Let  $f : X \rightarrow \mathfrak{R}$ , we say  $f$  is coercive if

$$\lim_{\|x\| \rightarrow \infty} f(x) = \infty \quad (17)$$

### Lemma 3.1

Let  $(X, \|\cdot\|)$  be a normed space and  $f : X \rightarrow \mathfrak{R}$ . Then  $\lim_{\|x\| \rightarrow \infty} f(x) = \infty$  if and only if  $[f \leq \lambda]$  is bounded for every  $\lambda \in \mathfrak{R}$  i.e  $f$  is coercive if and only if the level set is bounded.

Proof of Lemma 3.1:

Since  $f$  is coercive i.e  $\lim_{\|x\| \rightarrow \infty} f(x) = \infty$ ,  $\exists$  a scalar  $r > 0$  such that  $\|x\| > r, f(x) > \lambda \forall \lambda \in \mathfrak{R}$ .

$$\iff \lambda \in \mathfrak{R}, \exists r > 0 : [f \leq \lambda] \subset B(0, r) \quad (18)$$

This completes the proof.

### Theorem 3.1 ([11,p.100])

Let  $f \in \Gamma(X)$

- (i) If there exists  $\lambda > r(f, C)$  such that  $[f \leq \lambda]$  is  $w$ -compact, then  $S(f, C) \neq \phi$ .
- (ii) If  $X$  is a reflexive Banach space and  $f$  is coercive then  $S(f, C) \neq \phi$ .

**Proof of Theorem 3.1**

- (i) Of course,  $v(f, X) = v(f, [f \leq \lambda])$ . Since  $f$  is lower semicontinuous and convex,  $f$  is  $w$ -lsc. The conclusion follows using the Weierstrass theorem applied to the function  $f|_{[f \leq \lambda]}$ .
- (ii) Because  $f$  is coercive (see Lemma 3.1),  $[f \leq \lambda]$  is bounded for every  $\lambda \in \mathfrak{R}$  since  $[f \leq \lambda]$  is  $w$ -closed and  $X$  is reflexive, we have that  $[f \leq \lambda]$  is  $w$ -compact for every  $\lambda \in \mathfrak{R}$ . The conclusion follows from (i).

**Theorem 3.2**

Let  $D \subset X$  be a non-empty closed convex set and  $u_o \in X$

- (i) If  $X$  is a reflexive Banach space then  $P_D(u_o) \neq \phi$
- (ii) If  $X$  is a strictly convex normed space then  $P_D(u_o)$  has at most one element.

**Proof of Theorem 3.2**

- (i) Let us consider the function  $f := \| -u_o \| + l_D$ . Since  $f$  is Lipschitzian, then it is necessarily convex and lower semi continuous. By Theorem 3.1, there exists  $\bar{u} \in X$  such that  $f(\bar{u}) \leq f(u)$  for every  $u \in X$  i.e  $\bar{u} \in P_D(u_o)$ .
- (ii) This is a well known result and we omit the proof,(see[4]).

## 4 Concluding Remarks

In this paper, we investigated and formulated a Weber problem as an equivalent minimum norm problem. By the preceding sections, we have shown that the objective function is Lipschitzian and thus established the existence of global minimum norm solution using coercivity conditions.

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