A Note on Zero-dimensional Nearness Frames

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Abstract

We extend McKee's notion of 0-dimensional nearness spaces [6] to the frame counterparts, and then draw up a characterization of quotients of *fine*, 0-dimensional nearness frames.
Key words: frame, cover, nearness, zero-dimensionality.

1 Introduction

A *frame* L is a complete lattice satisfying the infinite distributive law: for any $a \in L$ and any $S \subseteq L$,

$$a \wedge \bigvee S = \bigvee \{a \wedge s \,|\, s \in S\}.$$

Thus one envisages a frame L as having the lattice structure $(L, \wedge, \bigvee, 0, 1)$, where 0 is the *bottom* element, and 1 the *top*. A *frame homomorphism* (or frame map) $f : L \longrightarrow M$, where L, M are frames, is a map preserving finite meets and arbitrary joins. In that case $f(0) = 0$ and $f(1) = 1$. We write $\mathbb{F}\mathrm{rm}$ for the category of frames and frame maps.

A typical example of a frame is a topology $\mathcal{O}(X)$ on a topological space X. And if $f : X \longrightarrow Y$ is a continuous map between topological spaces, then $f^{-1} : \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$ is a frame map. Clearly, this relationship is that of contravariance $\mathcal{O} : \mathbb{T}\mathrm{op} \longrightarrow \mathbb{F}\mathrm{rm}$. A standard reference for frames (dually called *locales*) is [5].

By a *cover* A of a frame L we mean a subset of L such that $\bigvee A = 1$. So $\mathrm{Cov}(L)$ becomes the set of all covers of the frame L. And the frame L is *compact* if for any $A \in \mathrm{Cov}(L)$, there is a finite $F \subseteq A$ in $\mathrm{Cov}(L)$. If $A, B \in \mathrm{Cov}(L)$, we say A *refines* B and write $A \leqslant B$ if for every $a \in A$, there is $b \in B$ such that $a \leqslant b$. We also define a collection $\mathrm{FCov}(L)$ of all covers of L refined by some finite cover.

The *star* of $x \in L$ with respect to a cover A is a member of L written

$$Ax = \bigvee \{a \in A \,|\, a \wedge x \neq 0\}.$$

Further we write $AB = \{Ax \mid x \in B\}$, which is a cover of L if A and B are both covers. We say A *star-refines* B, written $A \leqslant^* B$, if $AA \leqslant B$. In order to define a *nearness* we need the strongly below relation: For any collection $\mu \subseteq \mathrm{Cov}(L)$, we say $x \in L$ is *μ-strongly below* $y \in L$, written $x \lhd_\mu y$ (or simply $x \lhd y$), if there is a cover $A \in \mu$ such that $Ax \leqslant y$. The following properties of the relation $\lhd$ may be appropriate to mention here.

(1) If $x \lhd y,\ a \leqslant x$ and $y \leqslant b$, then $a \lhd b$.

(2) If $x \lhd y$ and $a \lhd b$, then $x \wedge a \lhd y \wedge b$ and $x \vee a \lhd y \vee b$.

(3) If $x \lhd_\mu y$, then there is $z \in L$ such that $x \lhd_\mu z \lhd_\mu y$ given μ is a *uniformity* on L.

We may now state the definition of nearness frames.

Definition 1 *A nonempty collection $\mu \subseteq Cov(L)$ is called a* nearness *on L if the following hold:*

n1. Whenever $A \in \mu$ refines $B \in Cov(L)$, then $B \in \mu$.

n2. Whenever $A,\ B \in \mu$, then $A \wedge B \in \mu$.

n3. Every $x \in L$ can be expressed as $x = \bigvee \{y \in L \mid y \lhd_\mu x\}$. (Admissibility property).

The pair (L, μ) is called a *nearness frame* whenever μ is a nearness on the frame L. Members of μ are called uniform covers, and $f : (L, \mu) \longrightarrow (M, \eta)$ is a uniform frame homomorphism between nearness frames if for every $A \in \mu,\ f(A) \in \eta$. We write $\mathbb{N}\mathbb{F}\mathrm{rm}$ for the category of nearness frames and uniform frame maps.

The concept of *nearness* in spaces was first introduced by H. Herrlich [3] as an axiomatization of the concept of nearness between arbitrary collections of sets. Such a development can be envisaged from the fact that one can obtain *topological spaces* via axiomatizing the concept of nearness between a point x and a set A - namely, the requirement that "x belongs to the closure of A". By further axiomatizing the concept of nearness between two sets one obtains *proximity spaces.* Also, what are dubbed *contiguity spaces* arise from an axiomatization of nearness between finite collections of sets. Hence, in that sense, nearness spaces evolved naturally.

2 0-dimensionality

Definition 2 *(i) A nearness frame (L, μ) is* <u>zero-dimensional</u> *(also written 0-dimensional) if for any cover $A \in \mu$, there is cover $B \in \mu$ refining A and with*

the property: for each $b \in B$, $\{b, b^*\} \in \mu$.
(ii) Let (L, μ) be 0-dimensional, and $U \in \mu$. We call $V \in \mu$ a witness for U if V refines U and for any $v \in V$, we have $\{v, v^*\} \in \mu$.

In retrospect, one may say that a nearness frame (L, μ) is 0-dimensional if each of its uniform covers has a witness. This definition extends McKee's notion of 0-dimensional nearness spaces [6], where a nearness space (X, ξ) is called 0-dimensional if for each cover $\mathcal{A} \in \xi$, there is a cover $\mathcal{B} \in \xi$ refining $\mathcal{A}$ and having the property that for any $B \in \mathcal{B}$, we have $\{B, X - B\} \in \xi$. A further observation in [6] is that 0-dimensional nearness spaces are regular. Consequently, employing the terminology of S.S. Hong and Y.K. Kim [4], 0-dimensional nearness spaces are *framed.* [Now if (X, ξ) is a nearness space with $x \in X$ and A, $B \subseteq X$, we recall that $x \lhd_\xi A$ iff $\{X - \{x\}, A\} \in \xi$ iff $x \in \mathrm{int}_\xi(A)$, and that $B \lhd_\xi A$ iff $\{X - B, A\} \in \xi$. We say that (X, ξ) is framed if whenever $x \lhd_\xi A$ in (X, ξ), there is $B \subseteq X$ such that $x \lhd_\xi B \lhd_\xi A$]. We further take note of the observation in [4] that a nearness space (X, ξ) is framed iff the family μ of open uniform covers in (X, ξ) is a nearness on the associated frame $\mathcal{O}(X)$ of open subsets of X.

Now if $V \in \mathcal{O}(X)$, with X being a topological space, then $V^* = \mathrm{int}(X - V)$ is the pseudocomplement of V in $\mathcal{O}(X)$. So a framed nearness space (X, ξ) is 0-dimensional iff the associated nearness frame $(\mathcal{O}(X), \mu)$ is 0-dimensional.

Recall the following notions:
(i) An element x in a frame L is said to be complemented if $x \vee x^* = 1$ (the property $x \wedge x^* = 0$ always holds for pseudocomplements).
(ii) A frame L is 0-dimensional if every $a \in L$ is a join of complemented members $b \leqslant a$ in L.
(iii) We say that a cover A of a frame L is a partition if every member of A is complemented and that whenever $x \neq y$ in A, then $x \wedge y = 0$.

We note that if each uniform cover A of a nearness frame (L, μ) is refined by a uniform partition P, then the nearness frame (L, μ) is 0-dimensional. For if P is a uniform partition, then for any $x \in P$, we have that P refines $\{x, x^*\}$, since whenever $a \neq x$ in P we have $a \wedge x = 0$, (so $a^* \vee x^* = 1$ and $a = a \wedge 1 = a \wedge (a^* \vee x^*) = (a \wedge a^*) \vee (a \wedge x^*) = 0 \vee (a \wedge x^*) = a \wedge x^*$) so that $a \leqslant x^*$.

Proposition 3 *If a nearness frame (L, μ) is 0-dimensional, then the underlying frame L is 0-dimensional.*

Proof: Let $a \in L - \{0\}$. Now $a = \bigvee\{x \in L \mid x \lhd_\mu a\}$ by property $n3$ for nearnesses. We need to show that a can be expressed as a join of fully complemented elements below it in L. Consider a typical $x \lhd_\mu a$. This means $\{x^*, a\} \in \mu$. And since (L, μ) is 0-dimensional, there is $B \in \mu$ refining $\{x^*, a\}$ and having the property that $\{b, b^*\} \in \mu$ for every $b \in B$.

Now we know, from properties of covers, that $x \leqslant Bx = \bigvee\{y \in B \mid y \wedge x \neq 0\}$. But $y \wedge x \neq 0$ implies that $y \nleqslant x^*$, so that $y \leqslant a$, since B refines $\{x^*, a\}$.

And each y is complemented, by property of B. So B has provided us with complemented elements y such that $x \leqslant \bigvee y \leqslant a$. Hence a is expressible as a join of complemented elements below it, so that L is 0-dimensional. ■

The notion of *topological* nearness space (which is a nearness space where each open cover is uniform) is essential in the main result of McKee's paper [6]. This notion has its analogue in *fine* nearness frames. Now (L, μ) is a fine nearness frame if $\mu = \mathrm{Cov}(L)$ (i.e μ consists of all covers). We have the result:

Proposition 4 *A fine nearness frame (L, μ) is 0-dimensional iff the underlying frame L is 0-dimensional.*

Proof: The implication $\Rightarrow$ follows from Proposition 3. As for the converse, suppose L is a 0-dimensional frame, and let $A \in \mu$. Then for each $a \in A$, put $B_a = \{x \in L \mid x \leqslant a \text{ and } x \text{ is complemented}\}$. Then the set $B = \bigcup_{a \in A} B_a$ is a witness for A. (Each $\{x, x^*\}$ is a uniform cover since (L, μ) is fine). Hence (L, μ) is 0-dimensional. ■

We also bring the notion of *strong nearness frame* into perspective here: (L, μ) is a *strong* nearness frame if for any $A \in \mu$, there is $B \in \mu$ such that $B \lhd_\mu A$. Here $B \lhd_\mu A$ means for any $b \in B$, there is $a \in A$ such that $b \lhd_\mu a$. In that case we say B *strongly refines* A. [Note that $B \lhd_\mu A$ implies $B \leqslant A$, since $b \lhd_\mu a$ implies $b \leqslant a$.]

Proposition 5 *Every 0-dimensional nearness frame is strong.*

Proof: Given (L, μ) 0-dimensional. Let $A \in \mu$ and B a witness for A. We show that B strongly refines A. Now, given $b \in B$, we have $a \in A$ such that $b \leqslant a$. Also $\{b, b^*\} \in \mu$. So $\{a, b^*\} \in \mu$. Therefore $b \lhd_\mu a$ so that B strongly refines A. ■

Let (L, μ) and (M, η) be nearness spaces, and $f : L \longrightarrow M$ a uniform map. We refer to the following terminology (according to [1]):
(i) f is a *surjection* if it is onto and $\eta = \{f(A) \mid A \in \mu\}$.
(ii) f is a *strict surjection* (or uniform *quotient* map) if it is a dense (i.e. $f(x) = 0$ implies $x = 0$) surjection and $\{f_*(C) \mid C \in \eta\}$ generates μ. In this case we refer to the nearness frame (M, η) as *a quotient* of (L, μ).
(Here $f_* : M \longrightarrow L$ is a map called the right adjoint of f, and is defined by $f_*(z) = \bigvee\{x \in L \mid f(x) \leqslant z\}$. As a consequence $f(x) \leqslant z$ iff $x \leqslant f_*(z)$.)
(iii) We will call (M, η) *subfine* [2] if it is a quotient of a fine nearness frame.
(iv) If any strict surjection $(L, \mu) \xrightarrow{h} (M, \eta)$ is an isomorphism, we say that (M, η) is a *complete* nearness frame.
(v) A nearness frame (L, μ) is a *completion* of (M, η) if $L \xrightarrow{f} M$ is a strict surjection and (L, μ) is complete.

Proposition 6 *A quotient of a 0-dimensional nearness frame is 0-dimensional.*

Proof: Using the above notation, suppose (L, μ) is 0-dimensional and (M, η) its quotient. Let $C \in \eta$. Then $f_*(C) \in \mu$ has a witness $B \in \mu$. This means $f(B)$ is a witness for C since f preserves pseudocomplements, being a dense onto map. ■

Lemma 7 *If (L, μ) and (M, η) are nearness frames and $f : L \longrightarrow M$ a dense surjection, then (L, μ) is 0-dimensional iff (M, η) is 0-dimensional.*

Proof: The implication $(\Rightarrow)$ follows from Proposition 6.
$(\Leftarrow)$ Suppose (M, η) is 0-dimensional. Let $A \in \mu$. Form the set $C = \{x \in M \mid f_*(x) \leqslant a, \text{ for some } a \in A\}$. To see that $C \in \eta$, since f is a surjection, there is $B \in \eta$ such that $f_*(B)$ refines A. So for $b \in B$, there is $a \in A$ such that $f_*(b) \leqslant a$. This means $b \in C$; (i.e. $B \subseteq C$), and so B refines C. Consequently $C \in \eta$. This implies $f_*(C) \in \mu$ refines A.

Now let $D \in \eta$ be a witness for C. First, $f_*(D)$ refines ($f_*(C)$ which refines) A. Second, for $f_*(d) \in f_*(D)$, we have $\{d, d^*\} \in \eta$. So $\{f_*(d), f_*(d^*)\} \in \mu$. And since $f_*(d^*) \leqslant (f_*(d))^*$, we have $\{f_*(d), (f_*(d))^*\} \in \mu$. Hence $f_*(D)$ is a witness for A. ■

Corollary 8 *A nearness frame is 0-dimensional iff its completion is 0-dimensional.*

Taking the notions *fine* and *subfine* to be analogies for 'topological' and 'subtopological' respectively, we now have a frame version of the main result of [6].

Theorem 9 *Let (M, η) be a nearness frame. Then the following are equivalent:*
(i) η is the nearness induced by a dense surjection $f : L \longrightarrow M$ with (L, μ) being a fine, 0-dimensional nearness frame.
(ii) (M, η) is subfine and 0-dimensional.
(iii) The completion of (M, η) is both fine and 0-dimensional.

Proof: (i)$\Rightarrow$(ii): By Proposition 5, (L, μ) is strong, so $f : L \longrightarrow M$ is a strict surjection. Then, by definition, (M, η) is subfine, and by Lemma 7, (M, η) is 0-dimensional.
(ii)$\Rightarrow$(iii): follows from Corollary 8
(iii)$\Rightarrow$(i): Trivial [replace (L, μ) with the completion of (M, η)]. ■

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