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On Generalized Köthe-Toeplitz Duals and Pre-Duals

A. A. Ansari and V. K. Chaudhry*
Department of Mathematics and Statistics
D.D.U. Gorakhpur University, Gorakhpur-273009, INDIA
e-mail : aaansari@in.com
*e-mail : vkc_ddumath@yahoo.co.in

Abstract

In this paper we generalized the notion of Köthe-Toeplitz duals and pre-duals of generalized sequence spaces on introducing the concept of $\eta-$dual of generalized sequence spaces.

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1. Introduction

Throughout the paper $s(X)$, $c(X)$, $c_0(X)$, $l_1(X)$, $l_p(X)$, $l_\infty(X)$ denote the space of all, convergent, null, absolutely summable, $p-$absolutely summable, bounded $X-$valued sequences respectively where X is a Banach space. In a similar way, replacing X by $B(X,Y)$ in the above where in $B(X,Y)$ denote the Banach space of bounded linear operators from the Banach space X to the Banach space Y, we can obtain the corresponding space of operators like $s(B(X,Y))$, $c(B(X,Y))$, $c_0(B(X,Y))$ etc. [7].

Let X and Y be Banach spaces over the field $\mathcal{C}$ of complex numbers and $B(X,Y)$ denote the Banach space of all bounded linear operators from X into Y with respect to supremum norm of operators i.e. if $T \in B(X,Y)$ the operator norm of T is

$$||T|| = \sup\{||T_x|| : x \in U\}$$

where $U = \{x \in X : ||x|| \leq 1\}$ is the closed unit sphere in X. X^* will denote the continuous dual of X. The zero element of $X, Y, B(X, Y)$ and X^* will denoted by 0 and the meaning will be clear by the context.

The Köthe-Toeplitz duals, the $\alpha-$ and $\beta-$duals, for the set $E(X)$ of $X-$valued sequences have been defined as

Definition (1.1). ([5]). Let X and Y be a Banach spaces and $\overline{A} = (A_k)$ a sequence of linear but not necessarily bounded operators A_k on X into Y. Suppose $E(X)$ is a non-empty subset of $s(X)$, then $\alpha-$dual of $E(X)$ is defined by

$$E^{\alpha}(X) = \{\overline{A} = (A_k) : \sum_{k=1}^{\infty} ||A_k x_k|| \text{ converges for all } \overline{x} = (x_k) \in E(X)\},$$

and $\beta-$dual of $E(X)$ is defined as

$$E^{\beta}(X) = \{\overline{A} = (A_k) : \sum_{k=1}^{\infty} A_k x_k \text{ converges for all } \overline{x} = (x_k) \in E(X)\},$$

where the convergence in $E^{\beta}(X)$ is with respect to norm of Y.

Robinson [1] considered the action of infinite matrices of linear operator from one Banach space into another on sequences of elements of the former Banach space. The $\alpha-$ and $\beta-$duals for various vector valued sequence spaces have been obtained in terms of sequence of operators ([5]).

Srivastava and Ansari [7] have introduced the concept of Köthe-Toeplitz pre-α dual and pre-β dual for the set $E(B(X, Y))$ of $B(X, Y)-$valued sequences which are defined as follows.

Definition (1.2). ([7]). Let X and Y be a Banach spaces and $E(B(X, Y))$ be a non-empty subset of $s(B(X, Y))$, then pre-α and pre-β dual are respectively defined as

$$E^{\alpha}(B(X,Y)) = \{\overline{x} = (x_k) : x_k \in X, \sum_{k=1}^{\infty} ||A_k x_k|| \text{ converges for all}$$

$$\overline{A} = (A_k) \in E(B(X,Y))\},$$

and

$$E^{\beta}(B(X,Y)) = \{\overline{x} = (x_k) : x_k \in X, \sum_{k=1}^{\infty} A_k x_k \text{ converges in } Y \text{ for all}$$

$$\overline{A} = (A_k) \in E(B(X,Y))\}.$$

The pre-α and pre-β duals of various set of $B(X,Y)$-valued sequences have been obtained in Srivastava and Ansari [6], [7].

We first define $\eta-$dual of $X-$valued and pre-η dual of $B(X,Y)$- valued sequences.

Definition (1.3). Let $E(X)$ be a non-empty subset of $s(X)$. Let $E(B(X,Y))$ be a non-empty subset of $s(B(X,Y))$ and let $r >$ (equal) 1. Then $\eta-$dual of $E(X)$ and pre-η dual of $E(B(X,Y))$ is defined respectively as

$$E^{\eta}(X) = \{\overline{A} = (A_k) : \sum_{k=1}^{\infty} ||A_k x_k||^r \text{ converges for all } \overline{x} = (x_k) \in E(X)\},$$

and

$$E^{\eta}(B(X,Y)) = \{\overline{x} = (x_k) : \sum_{k=1}^{\infty} ||A_k x_k||^r \text{ converges for all}$$

$$\overline{A} = (A_k) \in E(B(X,Y))\}.$$

Taking $r = 1$ in the above definition we get the $\alpha-$dual of $E(X)$ and pre-α dual of $E(B(X,Y))$.

We now state some known results in the form of Lemmas.

Lemma (1.4). (i) $E(X) \subset F(X)$ implies $E^\eta(X) \supset F^\eta(X)$ for every $E(X)$, $F(X) \subset S(X)$.

(ii) $E(B(X,Y)) \subset F(B(X,Y))$ implies $E^\eta(B(X,Y)) \supset F^\eta(B(X,Y))$ for every $E(B(X,Y))$, $F(B(X,Y)) \subset s(B(X,Y))$.

Proof. (i) Let $(A_k) \in F^\eta(X)$ and let $\overline{x} = (x_k) \in E(X)$. Since $E(X) \subset F(X)$, therefore $\overline{x} = (x_k) \in F(X)$, so

$$\sum_{k=1}^{\infty} ||A_k x_k||^r < \infty \Rightarrow (A_k) \in E^\eta(X).$$

(ii) Let $(x_k) \in F^\eta(B(X,Y))$ and let

$$\overline{A} = (A_k) \in E(B(X,Y)).$$

Since $E(B(X,Y)) \subset F(B(X,Y))$, therefore

$$\overline{A} = (A_k) \in F(B(X,Y))$$

So,

$$\sum_{k=1}^{\infty} ||A_k x_k||^r < \infty \Rightarrow \overline{x} = (x_k) \in E^\eta(B(X,Y)).$$

Lemma (1.5). (i) $c_0^\alpha(X) = c^\alpha(X) = l_\infty^\alpha(X) = l_1(B(X,Y))$,

(ii) $c_0^\alpha(B(X,Y)) = c^\alpha(B(X,Y)) = l_\infty^\alpha(B(X,Y)) = l_1(X)$.

2. Main Results

In this section we find the η-dual of some $X-$valued and pre-η dual of $B(X,Y)-$valued sequence spaces.

Proposition (2.1). $l_r^\eta(X) \subset l_\infty(B(X,Y))$.

Proof. The proof is by the method of contradiction. Suppose if possible that $(A_k) \in l_r^\eta(X)$ but $A_k \notin l_\infty(B(X,Y))$, i.e.

$$\sup_k ||A_k|| = \infty.$$

Then there exist a strictly increasing sequence of integer (k(i)) such that

$$||A_{k(i)}|| > i^s, \text{ for some fixed } s > 1,$$

and hence we can find a sequence (x_i) in U (the closed unit sphere in X) such that

$$||A_{k(i)}x_i|| > i^s. \tag{1}$$

Now, we define a sequence (z_k) by

$$z_k = \begin{cases} \dfrac{x_i}{i^s}, & \text{if } \ k = k(i) \\ 0, & \text{if } \ k \neq k(i) \end{cases} \tag{2}$$

Since

$$\begin{aligned} ||z_k|| &= \left|\left|\frac{x_i}{i^s}\right|\right| = \frac{||x_i||}{i^s} \qquad \text{for } k = k(i) \\ &\leq \frac{1}{i^s}, \qquad (\text{as} \quad x_i \in U), \qquad k = k(i), \end{aligned}$$

and zero eleswhere. Therefore $\sum_{k=1}^{\infty} ||z_k||^r < \infty$, i.e. $z_k \in l_r(X)$. But $\sum_{k=1}^{\infty} ||A_k x_k||^r$ diverges as

$$\begin{aligned} ||A_{k(i)}z_{k(i)}|| &= \frac{1}{i^s}||A_{k(i)}x_i|| \qquad &(\text{from 2}) \\ &> 1 \qquad &(\text{from 1}). \end{aligned}$$

This contradicts that $(A_k) \in l_r^\eta(X)$. Therefore $\sup_k ||A_k|| < \infty$ and $(A_k) \in l_\infty(B(X,Y))$. Therefore the assertion in the proposition is proved.

Proposition (2.2). $l_\infty(B(X,Y)) \subset l_r^\eta(X)$.

Proof. Let $(A_k) \in l_\infty(B(X,Y))$, therefore there exist a positive real number M such that

$$\sup_k ||A_k|| \leq M.$$

Now, if $\overline{x} = (x_k) \in l_r(X)$, then the assertion in the proposition follows from the following

$$\begin{aligned} \sum_{k=1}^{n} ||A_k x_k||^r &\leq \sum_{k=1}^{n} ||A_k||^r ||x_k||^r \\ &\leq \sup_{1\leq k<\infty} ||A_k||^r \sum_{k=1}^{\infty} ||x_k||^r \\ &\leq M^r \sum_{k=1}^{\infty} ||x_k||^r < \infty. \end{aligned}$$

On taking limit when $n \to \infty$, we get

$$\sum_{k=1}^{\infty} ||A_k x_k||^r < \infty.$$

By proposition 2.1 and 2.2 we get the following theorem.

Theorem (2.3). $l_r^\eta(X) = l_\infty(B(X,Y))$.

Proposition (2.4). $c_0^\eta(X) \subset l_r(B(X,Y))$

Proof. The proof is by the method of contradiction. Suppose if possible that $(A_k) \in c_0^\eta(X)$ but $(A_k) \notin l_r(B(X,Y))$, i.e.

$$\sum_{k=1}^{\infty} ||A_k||^r = \infty.$$

Then there exist a strictly increasing sequence (n_i) and a sequence (z_k) in U such that $2||A_k z_k|| > ||A_k||$ and

$$\sum_{k=n_i+1}^{n_{i+1}} ||A_k||^r > (2i)^r, \quad \text{for } i \in N.$$

Now, we define a sequence $\overline{x} = (x_k)$ as

$$x_k = \begin{cases} \dfrac{z_k}{i}, & \text{for } \; n_i < k \leq n_{i+1} \\ 0, & \text{for } \; k \leq n_1 \end{cases}$$

Then $x \in c_0(X)$ and

$$\sum_{n_i+1}^{n_{i+1}} ||A_k x_k||^r = \sum_{n_i+1}^{n_{i+1}} \frac{||A_k||^r}{(2i)^r} > 1,$$

which shows that $\sum\limits_{k=1}^{\infty} ||A_k x_k||^r$ is divergent. This contradicts the fact that $(A_k) \in c_0^{\eta}(X)$. Therefore the assertion in the proposition is proved.

Proposition (2.5). $l_r(B(X,Y)) \subset l_\infty^{\eta}(X)$.

Proof. Let $(A_k) \in l_r(B(X,Y))$. Then

$$\sum_{k=1}^{\infty} ||A_k||^r < \infty.$$

Now, if $\overline{x} = (x_k) \in l_\infty(X)$, then the assertion in the proposition follows from the following

$$\begin{aligned} \sum_{k=1}^{n} ||A_k x_k||^r &\leq \sum_{k=1}^{n} ||A_k||^r ||x_k||^r \\ &\leq \sup_{1 \leq k < \infty} ||x_k||^r \sum_{k=1}^{\infty} ||A_k||^r \\ &\leq \sum_{k=1}^{\infty} ||A_k||^r < \infty. \end{aligned}$$

On taking limit when $n \to \infty$, we get

$$\sum_{k=1}^{\infty} ||A_k x_k||^r < \infty.$$

Remark 1. We note that, since

$$c_0(X) \subset c(X) \subset l_\infty(X),$$

Therefore by lemma 1.4, we have

$$l_\infty^\eta(X) \subset c^\eta(X) \subset c_0(X).$$

When the inclusion relations are combined with proposition 2.4 and 2.5, we get the following theorem :

Theorem (2.6). $c_0^\eta(X) = c^\eta(X) = l_\infty^\eta(X) = l_r(B(X,Y))$.

Proposition (2.7). $l_r^\eta(B(X,Y)) \subset l_\infty(X)$.

Proof. Suppose if possible that $\overline{x} = (x_k) \in l_r^\eta(B(X,Y))$ but $\overline{x} = (x_k) \notin l_\infty(X)$, i.e.

$$\sup_k ||x_k|| = \infty.$$

Then there exist a subsequence of integers $(k(i))$ such that

$$||x_{k(i)}|| > i^{2/r}. \tag{3}$$

By Hahn-Banach theorem, each x_k determine an $f_k \in X^*$ with $||f_k|| < 1$, such that

$$f_k(x_k) = ||x_k||.$$

Now we define for, $x \in X$,

$$A_k(x) = \begin{cases} \dfrac{1}{i^{2/r}} f_k(x) y, & k = k(i) \\ 0, & k \neq k(i) \end{cases}$$

where $y \in Y$ with $||y|| = 1$, then

$$||A_{k(i)}|| = \sup_{||x|| \leq 1} \frac{1}{i^{2/r}} |f_k(x)|\ ||y||$$

$$= \frac{||f_k||}{i^{2/r}} \leq \frac{1}{i^{2/r}} \ .$$

Hence

$$\sum_{k=1}^{\infty} ||A_k||^r = \sum_{i=1}^{\infty} ||A_{k(i)}||^r < \sum_{i=1}^{\infty} \frac{1}{i^2} < \infty$$

which shows that $\overline{A} = (A_k) \in l_r(B(X,Y))$. But

$$||A_{k(i)} x_{k(i)}||^r = \frac{1}{i^2} |f_{k(i)}(x_{k(i)})|^r \ ||y||^r$$

$$= \frac{||x_{k(i)}||^r}{i^2} > 1.$$

which shows that $\sum\limits_{k=1}^{\infty} ||A_k x_k||^r$ is divergent. This contradicts that fact that $\overline{x} = (x_k) \in l_r^\eta(B(X,Y))$. Therefore the assertion in the proposition is proved.

Proposition (2.8). $l_\infty(X) \subset l_r^\eta(B(X,Y))$.

Proof. Let $\overline{x} = (x_k) \in l_\infty(X)$. Therefore there exist a positive real number M such that

$$\sup_k ||x_k|| < M.$$

Now, if $\overline{A} = (A_k) \in l_r(B(X,Y))$, then the assertion in the proposition follows from the following. For any positive integer n, we have

$$\sum_{k=1}^{n} ||A_k x_k||^r \leq \sum_{k=1}^{n} ||A_k||^r \ ||x_k||^r$$

$$\leq \sup_{1 \leq k < \infty} ||x_k||^r \sum_{k=1}^{\infty} ||A_k||^r$$

$$\leq M^r \sum_{k=1}^{\infty} ||A_k||^r < \infty.$$

On taking limit $n \to \infty$, we get

$$\sum_{k=1}^{\infty} ||A_k x_k||^r < \infty.$$

By proposition 2.7 and 2.8, we get the following theorem :

Theorem (2.9). $l_r^\eta(B(X,Y)) = l_\infty(X)$.

Proposition (2.10). $c_0^\eta(B(X,Y)) \subset l_r(X)$.

Proof. Let if possible $\overline{x} = x_k \in c_0^\eta(B(X,Y))$ but $\overline{x} = (x_k) \notin l_r(X)$ i.e.

$$\sum_{k=1}^{\infty} ||x_k||^r = \infty.$$

Then for the sequence of integer (i), there exist a sequence of integers (n_i) such that

$$\sum_{k=n_i+1}^{n_{i+1}} ||x_k||^r > i^r.$$

By the Hahn Banach theorem, for each $x_k \in X$, there exist $f_k \in X^*$ such that

$$||x_k|| = f_k(x_k) \quad \text{and} \quad ||f_k|| \leq 1.$$

Now we define a sequence of operators $\overline{A} = (A_k)$ as, for $x \in X$,

$$A_k(x) = \begin{cases} \frac{1}{i} f_k(x) y, & \text{for } \mathrm{n_i < k \leq n_{i+1}} \\ 0 & \text{for } \mathrm{k \leq n_1} \end{cases}$$

where $y \in X$, and $||y|| = 1$.

Since

$$\begin{aligned} ||A_k|| &= \sup ||A_k(x)||, \quad ||x_k|| \leq 1 \\ &= \frac{1}{i} \, ||f_k(x) y||, \qquad n_i < k \leq n_{i+1}, \; ||x_k|| \leq 1 \\ &= \frac{1}{i} ||f_k|| < \frac{1}{i} \, , \qquad n_i < k \leq n_{i+1} \end{aligned}$$

and $||A_k|| = 0$, eleswhere.

Therefore $A_k \to 0$ as $k \to \infty$ i.e. $A_k \in c_0(B(X,Y))$. But

$$\begin{aligned}\sum_{n_i+1}^{n_{i+1}} ||A_k x_k||^r &= \frac{1}{i^r} \sum_{n_i+1}^{n_{i+1}} ||f_k(x_k)y||^r \\ &= \frac{1}{i^r} \sum_{n_i+2}^{n_{i+1}} ||x_k||^r > 1\end{aligned}$$

which shows that $\sum_{k=1}^{\infty} ||A_k x_k||^r$ is divergent. This contradicts the fact that $\overline{x} = (x_k) \in c_0^\eta(B(X,Y))$. Therefore the assertion in the proposition is proved.

Proposition (2.11). $l_r(X) \subset l_\infty^\eta(B(X,Y))$.

Proof. Let $\overline{x} = (x_k) \in l_r(X)$. Then

$$\sum_{k=1}^{\infty} ||x_k||^r < \infty.$$

Now if $\overline{A} = (A_k) \in l_\infty(B(X,Y))$, then the assertion in the proposition follows from following. For any positive integer n, we have

$$\begin{aligned}\sum_{k=1}^{n} ||A_k x_k||^r &\leq \sum_{k=1}^{\infty} ||A_k||^r \; ||x_k||^r \\ &\leq \sup_{1\leq k<\infty} ||A_k||^r \sum_{k=1}^{\infty} ||x_k||^r \\ &\leq \sum_{k=1}^{\infty} ||x_k||^r < \infty.\end{aligned}$$

Remark 2. We note that, since

$$c_0(B(X,Y)) \subset c(B(X,Y)) \subset l_\infty(B(X,Y)),$$

therefore by lemma 1.4, we have

$$l_\infty^\eta(B(X,Y)) \subset c^\eta(B(X,Y)) \subset c_0^\eta(B(X,Y)).$$

When the inclusion relation are combined with proposition 2.10 and 2.11, we get following theorem :

Theorem (2.12). $c_0^\eta(B(X,Y)) = c^\eta(B(X,Y)) = l_\infty^\eta(B(X,Y)) = l_r(X)$.

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