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Finite Difference Method for Beam Equation with Free Ends Using Mathematica

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Abstract

For simple data functions $f(x,u)$ and $r(x)$, the exact solution $u(x)$ of the Euler-Bernouli's beam equation

$$\frac{d^2}{dx^2}[r(x)\frac{d^2u}{dx^2}] = f(x,u), \qquad 0 \leq x \leq L, \tag{1}$$

with the free ends boundary conditions

$$\begin{aligned} u(0) &= a_0, \qquad \frac{d^2u(0)}{dx^2} = b_0, \\ u(L) &= a_L, \qquad \frac{d^2u(L)}{dx^2} = b_L, \end{aligned} \tag{2}$$

can be found by standard methods well known in literature of ordinary differential equations and their applications. For more developed data functions, when exact methods fail, numerical methods are successfully applied to find an approximate solution of a broad class of the boundary value problems. In the paper, we present effective algorithms based on finite difference methods for linear and non-linear beam equations. We give the complete analysis of convergence of the algorithms with global error estimates. For the purpose of illustration of the methods and their applications, we design *Mathematica* modulae and solve a number of beam equations . The numerical solutions confirm effectiveness of the algorithms.

Keywords: Beam Equations, Finite Difference Methods [1]

1 Introduction

Beam equations have a long history starting from Leonardo da Vinci (1452-1519) and Galileo Galilee (1584-1642) developed by Leonard Euler (1707-1783), Jacob (1654-1705) and Daniel Bernoulli (1700-1782) in the eighteenth century. There are numerous structures that evidently were constructed through application of the beam theory. Practical applications of the beam equations

[1] This is a part of Ms. Thankane's Master Dissertation

are presented in broad literature (cf. [1],[2],[3],[7],[8]). These are very huge buildings like the Manhattan towers, long bridges across big rivers, aero planes and cars. In these structures, the beams are either used as supporting structures of the floor or as axles of cars. These would not be possible if it was not for the usage of the beam theory.
In this paper, we consider numerical solutions of the beam equations with use of Mathematica system for doing Mathematics (see [6]). The paper is intended to present a complete mathematical analysis of effective algorithms based on the Finite Difference Method illustrated by a number of examples. The paper can be also of an academic and scientific interest for those who deal with the beam equations and their applications including engineering theory and construction.
Boundary Value Problem. We consider the beam equation

$$\frac{d^2}{dx^2}[r(x)\frac{d^2u}{dx^2}] = f(x,u), \qquad 0 \le x \le L, \tag{3}$$

with the free ends boundary conditions

$$\begin{aligned} u(0) = a_0, \qquad \frac{d^2u(0)}{dx^2} = b_0, \\ u(L) = a_L, \qquad \frac{d^2u(L)}{dx^2} = b_L, \end{aligned} \tag{4}$$

Let us write the equation in the equivalent form:

$$\begin{aligned} \frac{d^2v}{dx^2} &= f(x,u), \\ r(x)\frac{d^2u}{dx^2} &= v, \qquad 0 \le x \le L \end{aligned} \tag{5}$$

with the corresponding boundary conditions

$$\begin{aligned} u(0) = a_0, \qquad v(0) = b_0, \\ u(L) = a_L, \qquad v(L) = b_L, \end{aligned} \tag{6}$$

1.1 Engineering Interpretation

A beam is common element of a structure of bridges, towers, buildings, etc.
In the linear case, when $f(x,u) = q(x)u + p(x)$, $r = EI$, E is the Young's modulus, and I is the area moment of inertia of the beam's cross section, the beam equation, takes the following form:

$$\frac{d^2}{dx^2}[E\,I\frac{d^2u}{dx^2}] = q(x)u + p(x), \qquad 0 \le x \le L, \tag{7}$$

Here, $u(x)$ is a deflection of the beam, $q(x)$ is the coefficient of ground elasticity, and $p(x)$ is a load force normal to the beam at the point x.

In the case, when $q(x) = 0$, and the beam is supported with free ends, the solution $u(x)$ of the boundary problem

$$\begin{aligned} &\frac{d^2}{dx^2}[E\, I\frac{d^2u}{dx^2}] = p(x), \qquad && 0 \le x \le L, \\ &u(0) = a_0, \qquad && \frac{d^2u(0)}{dx^2} = b_0, \\ &u(L) = a_L, \qquad && \frac{d^2u(L)}{dx^2} = b_L. \end{aligned} \tag{8}$$

describes deflection of the beam from the original position under the force of load $p(x)$. (see Figure 1 and Figure 2).

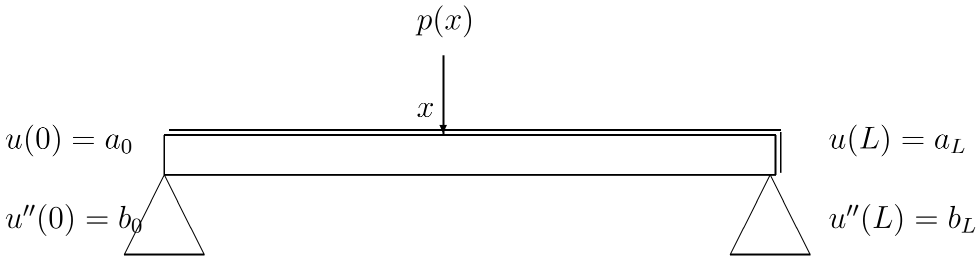

Figure. 1 Beam with free ends

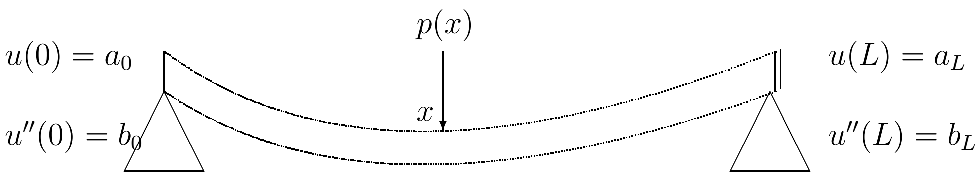

Figure. 2. Beam deflection

For simple data functions $f(x, u)$ and $r(x)$, the exact solution $u(x)$ of equation (3) with the boundary conditions can be found by standard methods well known in literature of ordinary differential equations and their applications (cf. [3]). For more developed data functions, when exact methods fail, numerical methods can be successfully applied to find an approximate solution of a broad class of the boundary value problems. We shall present some effective numerical methods for solving the boundary problems in the next sections.

We begin to consider the case, when the product of Young's Modulus E and the area moment of inertia I are constant. Then the equation (3) can be written as

$$\frac{d^4u}{dx^4} = \frac{1}{EI} f(x, u).$$

2 Finite Difference Scheme

We apply notation

$$x_i = ih, \quad h = \frac{L}{n+1}, \qquad u_i = u(x_i),$$
$$\frac{d^2u(x_i)}{dx^2} = \Lambda u(x_i,) + \psi_i^0(h), \qquad \frac{d^2v(x_i)}{dx^2} = \Lambda v_i + \psi_i^1(h)$$

for $i = 1, 2, ..., n$, where the finite difference operator

$$\Lambda u_i = \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2},$$

and the local truncation errors

$$\psi_i^0(h) = -\frac{h^2}{12}\frac{d^4u(\xi_i)}{dx^4}, \qquad \psi_i^1(h) = -\frac{h^2}{12}\frac{d^4v(\xi_i)}{dx^4} \tag{9}$$

for a certain $\xi_i \in (x_{i-1}, x_{i+1})$, provided that $u(x)$ is six times continuously differentiable in the interval $[0, L]$.
Now let us consider the case when $r(x) = 1$ starting with the standard finite difference approximation of the derivatives $\frac{d^2u}{dx^2}$ and $\frac{d^4u}{dx^4}$, on the mesh points

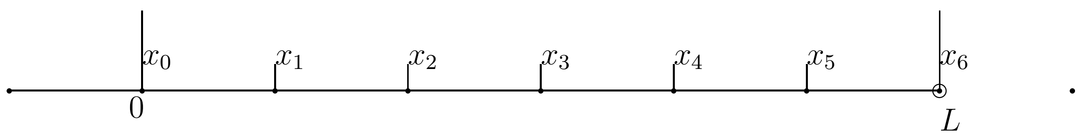

Figure 3. The knots $x_i,\ i = 1, 2, ..., n,\ n = 5$

Then, the standard finite difference scheme which approximates the two equations (5) with boundary conditions (6) is

$$\begin{aligned} &\Lambda v_i = f(x, u_i) + \psi_i^0(h), \\ &\Lambda u_i = v_i + \psi_i^1(h), \qquad\qquad i = 1, 2, 3, ..., n, \end{aligned} \tag{10}$$

with the truncation error $\psi_i(h) = O(h^2)$

2.1 Matrix Form of the Finite Difference Scheme

Let us write equations (10) in the form of the system of equations including the boundary value conditions.
Canceling the local truncation error $\psi_i(h)$ in (10), we arrive at the finite dif-

ference scheme in the form of the system of equations

$$
\begin{aligned}
&v_0 = b_0 \\
&\frac{v_{i-1} - 2v_i + v_{i+1}}{h^2} = f(x_i, u_i) \quad i = 1, 2, ..., n, \\
&v_{n+1} = b_L
\end{aligned} \tag{11}
$$

and

$$
\begin{aligned}
&u_0 = a_0 \\
&\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} = v_i, \qquad i = 1, 2, ..., n, \\
&u_{n+1} = a_L
\end{aligned} \tag{12}
$$

Eliminating v_0, v_{n+1} and $u_0,\ u_{n+1}$, we obtain

$$
\begin{aligned}
&\frac{2v_1 - v_2}{h^2} = b_0 - f(x_1, u_1) \\
&\frac{-v_{i-1} + 2v_i - v_{i+1}}{h^2} = -f(x_i, u_i) \quad i = 2, 3, ..., n-1, \\
&\frac{-v_{n-1} + 2v_n}{h^2} = b_L - f(x_n, u_n)
\end{aligned} \tag{13}
$$

and

$$
\begin{aligned}
&\frac{2u_1 - u_2}{h^2} = a_0 - v_1 \\
&\frac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} = -v_i, \qquad i = 2, 3, ..., n-1, \\
&\frac{-u_{n-1} + 2u_n}{h^2} = a_L - v_n
\end{aligned} \tag{14}
$$

Now, let us write the finite difference schemes (13) and (14) in the matrix form for the vector-solutions $U_h = \{u_1, u_2, ..., u_n\}$ and $V_h = \{v_1, v_2, ..., v_n\}$

$$M_0\, V_h = F(U_h), \qquad M_0\, U_h = F_h(V_h), \tag{15}$$

where

$$
M_0 = \frac{1}{h^2}\begin{bmatrix}
2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2
\end{bmatrix},
$$

and right sides

$$F_h(U_h) = \begin{Bmatrix} b_0 - f(x_1, u_1) \\ -f(x_2, u_2) \\ -f(x_3, u_3) \\ \vdots \\ -f(x_{n-1}, u_{n-1}) \\ b_L - f(x_n, u_n) \end{Bmatrix}, \qquad F_h(V_h) = \begin{Bmatrix} a_0 - v_1 \\ -v_2 \\ -v_3 \\ \vdots \\ -v_{n-1} \\ a_L - v_n \end{Bmatrix},$$

2.2 Convergence

Let us note that the matrix M_0 in the systems of equations (15) is positive definite. Indeed, the matrix has the eigenvalues (cf. [4], p.55):

$$\lambda_k = \frac{4}{h^2} \sin^2 \frac{k\pi}{2n}, \qquad k = 1, 2, ..., n$$

The eigenvalues $\lambda_k,\ k = 1, 2, ..., n$, satisfy the inequalities (cf. [4], p. 57)

$$\lambda_{n-2} > \lambda_{n-3} > \cdots > \lambda_1 > \frac{8}{L^2} \tag{16}$$

The following theorem holds (cf. [4], p. 59) :

Theorem 1 *A symmetric matrix M_0 which has positive eigenvalues is positive definite*

So, there exists a constant γ such that

$$(M_0 X, X) \geq \gamma(X, X)$$

for any vector $X \in R^n$ and $\gamma = \dfrac{8}{L^2}$

The inverse matrix M_0^{-1} is also positive definite and its spectral norm $||M_0^{-1}|| \leq \dfrac{1}{\gamma}$.

In conclusion, the norm of the matrix M_0^{-1} satisfies the inequality

$$||M_0^{-1}|| \leq \frac{L^2}{8}. \tag{17}$$

Global error estimate. Let $U,\ V$ be the exact solution and $U_h,\ V_h$ the approximate solution. Then, the global error $E_h(U) = U_h - U,\ E_h(V) = V_h - V$ satisfies the system of equations

$$M_0\ E_h(U) = \Psi_h(U), \qquad M_0\ E_h(V) = \Psi_h(V),$$

where, by (9), the local truncation error

$$||\Psi_h(U)|| \leq C\ h^2, \qquad ||\Psi_h(V)|| \leq C\ h^2$$

for a constant C independent of h.
So that

$$E_h(U) = M_0^{-1}\Psi_h(U), \qquad E_h(U) = M_0^{-1}\Psi_h(V).$$

Hence, we obtain the global error estimate

$$||U_h - U|| \leq K\ h^2, \qquad ||V_h - U|| \leq K\ h^2,$$

for a constant K independent of h.
Thus, we can state the following Theorem:

Theorem 2 *The method is convergent as fast as $O(h^2) -> 0$ when $h -> 0$.*

2.3 Homogeneous Boundary Conditions

In applications, it is convenient to transform the non-homogeneous boundary conditions to homogeneous one. We make the transformation by the mapping

$$u(x) = u^0(x) + w(x), \quad 0 \leq x \leq L,$$

where the interpolating polynomial $w(x)$ satisfies the boundary conditions

$$w(0) = a_0, \quad w(L) = b_0,$$

$$w''(0) = a_L, \quad w''(L) = b_L$$

To find the interpolating polynomial, we apply the *Mathematica* commands (cf. [6])

```
Enter
          n  (for example n=9 to be h=10;)
          p = {{0, {a_0, 0, b_0}}, {1, {a_L, 0, b_L}}};
```

Execute

```
          Simplify[InterpolatingPolynomial[p, x]];
```

to obtain

$$\begin{aligned} w(x) &= (a_0 + x^2((b_0/2 + ((-a_0 + a_L - b_0/2 + 1/2((6a_0 - 6a_L + b_0 \\ &- ((12a_0 - 12a_L + b_0 - b_L))((-1 + x))))((-1 + x)))x)) \end{aligned}$$

The new unknown $u^0(x)$ is the solution of the equation

$$\frac{d^4u^0}{dx^4} = f(x, u^0 + w) - \frac{d^4w}{dx^4}, \qquad 0 \leq x \leq L \tag{18}$$

and satisfies the homogeneous boundary conditions

$$u^0(0) = 0, \quad u^0(L) = 0,$$
$$u_0^{''}(0) = 0, \quad u_0^{''}(L) = 0.$$

We split equation (18) into two equations of the second order

$$\begin{aligned} \frac{d^2v^0}{dx^2} &= f(x, u^0 + w) - \frac{d^4w}{dx^4}, \\ \frac{d^2u^0}{dx^2} &= v^0(x), \qquad\qquad 0 \le x \le L \end{aligned} \tag{19}$$

with the corresponding homogeneous boundary conditions

$$\begin{aligned} u^0(0) = 0, \qquad v(0) = 0, \\ u^0(L) = 0, \qquad v(L) = 0, \end{aligned} \tag{20}$$

In the case of homogeneous boundary value conditions, we have the system of equations

$$\begin{aligned} & v_0^0 = 0 \\ & \frac{-v_{i-1}^0 + 2v_i^0 - v_{i+1}^0}{h^2} = -f(x_i, u_i^0 + w_i) + \frac{d^4w_i}{dx^4}, \quad i = 1, 2, ..., n, \\ & v_{n+1}^0 = 0 \end{aligned} \tag{21}$$

and

$$\begin{aligned} & u_0^0 = 0 \\ & \frac{-u_{i-1}^0 + 2u_i^0 - u_{i+1}^0}{h^2} = -v_i^0, \qquad\quad i = 1, 2, ..., n, \\ & u_{n+1}^0 = 0 \end{aligned} \tag{22}$$

Because $v_0^0 = 0, v_{n+1}^0 = 0$ and $u_0^0 = 0,\ u_{n+1}^0 = 0$, the system of equations takes the following form:

$$\begin{aligned} & \frac{2v_1^0 - v_2^0}{h^2} = -f(x_1, u_1^0 + w_1) + \frac{d^4w_1}{dx^4} \\ & \frac{-v_{i-1}^0 + 2v_i^0 - v_{i+1}^0}{h^2} = -f(x_i, u_i^0 + w_i) + \frac{d^4w_i}{dx^4}, \quad i = 2, 3, ..., n-1, \\ & \frac{-v_{n-1}^0 + 2v_n^0}{h^2} = -f(x_n, u_n^0 + w_n) + \frac{d^4w_n}{dx^4} \end{aligned} \tag{23}$$

and

$$\begin{aligned}
&\frac{2u_1^0 - u_2^0}{h^2} = -v_1^0 \\
&\frac{-u_{i-1}^0 + 2u_i^0 - u_{i+1}^0}{h^2} = -v_i^0, \qquad i = 2, 3, ..., n-1, \\
&\frac{-u_{n-1}^0 + 2u_n^0}{h^2} = -v_n
\end{aligned} \tag{24}$$

Now, let us write the finite difference schemes (23) and (24) for the solutions $U_h^0 = \{u_1^0, u_2^0, ..., u_n^0\}$ and $V_h^0 = \{v_1^0, v_2^0, ..., v_n^0\}$ in the matrix form

$$M_0\, V_h^0 = F(U_h^0), \qquad M_0\, U_h^0 = F_h(V_h^0), \tag{25}$$

where the matrix M_0 and the vectors $F(U_h^0),\ F(V_h^0)$ are given by formula (15) for $a_0 = a_L = 0$ and $b_0 = b_L = 0$

Example 1 *Consider the beam equation*

$$\frac{d^4 u}{dx^4} = \pi^4 x \sin \pi x - 4\pi^3 \cos \pi x, \qquad 0 \le x \le 1, \tag{26}$$

with the corresponding non-homogeneous boundary conditions

$$\begin{aligned}
u(0) = 0, &\qquad u''(0) = 2\pi, \\
u(1) = 0, &\qquad u''(1) = -2\pi,
\end{aligned} \tag{27}$$

Let us note that $u(x) = x \sin \pi x$ is the exact solution of the boundary problem. In order to transform the non-homogeneous boundary conditions to homogeneous one, we introduce the new unknown $u^0(x)$ by the substitution

$$u(x) = u^0(x) + w(x), \qquad 0 \le x \le 1,$$

where the interpolating polynomial $w(x)$ is obtained by the *Mathematica* command:

Enter

$$p = \{\{0, \{0, 0, 2\pi\}\}, \{1, \{0, 0, -2\pi\}\}\};$$

Execute

```
Simplify[InterpolatingPolynomial[p, x]];
```

to get

$$w(x) = -\pi(x-1)^2 x^2 (2x-1).$$

Then, the new unknown $u^0(x) = x \sin \pi x - w(x)$ satisfies the equations

$$\frac{d^2}{dx^2}\frac{d^2 u^0}{dx^2} = \pi^4 u^0 - 4\pi^3 \cos \pi x - \frac{d^2}{dx^2}\frac{d^2 w}{dx^2}, \qquad 0 \le x \le 1, \tag{28}$$

with the free ends homogeneous boundary conditions

$$\begin{aligned} u^0(0) &= 0, \qquad \frac{d^2u^0(0)}{dx^2} = 0, \\ u^0(1) &= 0, \qquad \frac{d^2u^0(1)}{dx^2} = 0, \end{aligned} \tag{29}$$

or in the form of two equations of the second order

$$\begin{aligned} \frac{d^2v^0}{dx^2} &= \pi^4 u^0 - 4\pi^3 \cos \pi x - \frac{d^2}{dx^2}\frac{d^2w}{dx^2}, \qquad 0 \le x \le 1, \\ \frac{d^2u^0}{dx^2} &= v^0, \qquad\qquad 0 \le x \le 1 \end{aligned} \tag{30}$$

with the corresponding homogeneous boundary conditions

$$\begin{aligned} u^0(0) &= 0, \qquad v(0) = 0, \\ u^0(1) &= 0, \qquad v(1) = 0. \end{aligned} \tag{31}$$

Then, the solution with the non-homogeneous boundary conditions

$$u(x) = u^0(x) + w(x), \qquad 0 \le x \le 1,$$

where the solution $u^0(x)$ with homogeneous boundary conditions is determined by the following algorithm.

3 Optimal Algorithm

In order to find the solution of equations (25), we shall modify the algorithm given in (cf. [5], p.165), which contains proportional number of arithmetic operations to the number of unknowns n. Let us note that a stable algorithm for solving a system of equations $MV = F$ with a square matrix M of dimension $n \times n$ and with n unknowns is called optimal if it solves the system of equations with the cost of the number of arithmetic operations proportional to n. In this sense the proposed algorithms are optimal. By the algorithm one can find a solution of a beam equation in the case of a function $f(x, u) = p(x)$ independent of u. In the next sections, we shall use the algorithm for any function $f(x, u)$.

Algorithm

$$
\begin{array}{llll}
To\ evaluate & V_h = (v_1, v_2, ..., v_n) & & \\
Set: & \alpha_1 = -\dfrac{1}{2}, & \beta_1 = \dfrac{F_1(V_h)h^2}{2} & \\
evaluate & \alpha_i = \dfrac{-1}{2+\alpha_{i-1}} & for\ \ i = 2, 3, \ldots, n-1, & \\
evaluate, & \beta_i = \dfrac{F_i(V_h)h^2 + \beta_{i-1}}{2+\alpha_{i-1}} & for\ \ i = 2, 3, \ldots, n, & \\
set\ : & v_n = \beta_n & & \\
evaluate\ : & v_i = \beta_i - \alpha_i v_{i+1} & for\ \ i = n-1, n-2, \ldots, 1 & \\
To\ evaluate & U_h = (u_1, u_2, ..., u_n) & & \\
Set: & \beta_1 = \dfrac{F_1(U_h)h^2}{2} & & \\
evaluate, & \beta_i = \dfrac{F_i(U_h)h^2 + \beta_{i-1}}{2+\alpha_{i-1}} & for\ \ i = 2, 3, \ldots, n, & \\
set\ : & u_n = \beta_n & & \\
evaluate\ : & u_i = \beta_i - \alpha_i u_{i+1} & for\ \ i = n-1, n-2, \ldots, 1 &
\end{array}
\tag{32}
$$

3.1 Mathematica Module *solveFreeEnds*

Applying the algorithm, we design *Mathematica* module *solveFreeEnds*
The module solves a beam equation with free ends under homogeneous boundary conditions.

The entry date:

- n is the number of interior mesh points in the interval $[0, L]$.
- $q(x)$ is a given continuous function.

Mathematica **module** *solveFreeEnds*

```
solveFreeEnds[q_, L_, n_] := Module[{x },
    h = N[L/(n + 1)];
    solveTri[g_] := Module[{al, be, x, fv, v1, u1},
       al[1] = -1/2;
       al[i_] := al[i] = -1/(2 + al[i - 1]);
       be[1] = g[[1]]h^2/2;
       be[i_] := be[i] = (g[[i ]]h^2 + be[i - 1])/(2 + al[i - 1]);
```

```
    x[n] = be[n];
    x[i_] := x[i] = be[i] - al[i]*x[i + 1];
    Table[x[i], {i, 1, n}]
    ];
fv = Table[q[i h], {i, n}];
v1 = solveTri[fv];
u1 = solveTri[v1];
u1
]
```

In order to invoke the module, we enter data

`q[x_],   n,   L,`

and execute the command

`solveFreeEnds[q_,n_,L_]`

As the result of execution of the module, we obtain the numerical solution in the form of the table at the mesh points

$$x_0 = 0,\ x_0 + h,\ x_0 + 2h, ..., x_0 + nh,\ x_{n+1} = L.$$

Example 2 *Find the solution $u(x)$ of the beam equation*

$$\frac{d^2}{dx^2}[\frac{d^2u}{dx^2}] = \pi^4 x \sin \pi x - 4\pi^3 \cos \pi x \qquad 0 \leq x \leq 1,$$

which satisfies the free ends boundary value conditions

$$u(0) = u(1) = 0,\ \ u''(0) = 2\pi \quad u''(1) = -2\pi.$$

First, we transform the boundary value problem with the non-homogeneous boundary conditions to the boundary value problem with the homogeneous boundary conditions (see the example (26)).
Then, we solve the beam equation

$$\begin{aligned} \frac{d^4u^0}{dx^4} &= \pi^4 x \sin \pi x - 4\pi^3 \cos \pi x \\ &+ 96\pi(x-1) + 12\pi(8x + 2(2x-1)), \qquad 0 \leq x \leq 1, \end{aligned} \tag{33}$$

with the corresponding homogeneous boundary conditions

$$\begin{aligned} u^0(0) = 0, &\qquad \frac{d^2u^0(0)}{dx^2} = 0, \\ u^0(1) = 0, &\qquad \frac{d^2u^0(1)}{dx^2} = 0. \end{aligned} \tag{34}$$

In order to apply the module, we define in *Mathematica* the function

$$q[x\_] := \pi^4 x \sin \pi x - 4\pi^3 \cos \pi x + 96\pi(x-1) + 12\pi(8x + 2(2x-1)),$$

and invoke the module for the number of interior mesh points $n = 9$ in the interval $[0, L] = [0, 1]$ with step size $h = \dfrac{L}{n+1} = 0.1$

```
solveFreeEnds[q,9,1]
```

Then, we write the solutions $u_h^0(x)$ and $u^0(x)$ rounded off to four decimal places in the table

x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$u_h^0(x)$	0.0069	0.0657	0.1864	0.3477	0.5083	0.6191	0.6361	0.5319	0.3072
$u^0(x)$	0.0105	0.0693	0.1873	0.3442	0.5	0.6068	0.6217	0.5185	0.2985

Table 1

The exact solution

$$u^0(x) = x \sin[\pi x] - w[x],$$

where

$$w(x) = -\pi(x-1)^2 x^2 (2x-1).$$

The absolute global error

$$globalError \;=\; \max_{1 \le i \le 9} |u_h^0(x_i) - u^0(x_i)| \;=\; 0.0143432$$

and relative error

$$relativeError \;=\; 1.43432\%$$

The exact solution $u(x) = x \sin \pi x$ satisfies the beam equation

$$\frac{d^4 u}{dx^4} \;=\; \pi^4 x \sin \pi x - 4\pi^3 \cos \pi x, \qquad 0 \le x \le 1, \tag{35}$$

and the non-homogeneous boundary value conditions

$$\begin{aligned} u(0) = 0, \qquad \frac{d^2 u(0)}{dx^2} = 2\pi, \\ u(1) = 0, \qquad \frac{d^2 u(1)}{dx^2} = -2\pi. \end{aligned} \tag{36}$$

Then, we get the solution $u_h(x) = u_h^0(x) + w(x)$.

3.2 Implicit iterative method

We shall solve the system of equations (25) by the following implicit iterative method:

$$\begin{aligned} &V_h^0 \quad \textit{given initial values} \\ &M_0\, U_h^{(m+1)} = F_h(V_h^m), \\ &M_0\, V_h^{(m+1)} = F_h(U_h^m), \qquad m = 0, 1, ... \end{aligned} \tag{37}$$

The following theorem holds:

Theorem 3 *The implicit iterative method is convergent for any given initial vector* $V_h^{(0)}$, *when* $L < 2\sqrt{2}$.

Proof. The difference of two following iterations

$$E^{(m+1)}(U_h) = U_h^{(m+1)} - U_h^{(m)}$$

satisfies the relation

$$\begin{aligned} M_0\, E^{(m+1)}(U_h) &= F(V_h^{(m+1)}) - F(V_h^{(m)}) \\ &= V_h^{(m+1)} - V_h^{(m)} = E^{(m+1)}(V_h), \end{aligned}$$

Hence, we have the recursive relation

$$E^{(m+1)}(U_h) = M_0^{-1} E^{(m)}(V_h), \qquad m = 0, 1, 2, ...,$$

Because $||M_0^{-1}|| \leq \frac{L^2}{8}$, therefore, by the above relation, we get the estimate

$$||E^{(m+1)}(U_h)|| \leq \frac{1}{\gamma} ||\, E_h^{(m)}(V_h)|| \leq \frac{1}{\gamma^m}\, ||\, E_h^{(0)}(V_h)||, \qquad m = 0, 1, 2, ..., \tag{38}$$

For $\gamma = \frac{8}{L^2}$, we have $\frac{1}{\gamma} = \frac{L^2}{8} < 1$, when $L < 2\sqrt{2}$.
Thus, by inequality (38), we conclude that

$$||E^{(m+1)}|| - > 0 \quad \text{, when} \quad m - > \infty,$$

since then $\frac{1}{\gamma^m} - > 0$. End of the proof.

3.3 Mathematica Module `solveFreeEndsImplicit`

We solve equation (25) with free ends, by the Implicit Iterative Method using the *Mathematica* module

```
solveFreeEndsImplicit[v0_, g_, L_, n_] := Module[{x  },
    h = L/(n + 1);
    solveTri[f_] := Module[{al, be, x},
       al[1] = -1/2;
       al[i_] := al[i] = -1/(2 + al[i - 1]);
       be[1] = f[[1]]/2;
       be[i_] := be[i] = (f[[i]] + be[i - 1])/(2 + al[i - 1]);
       x[n] = be[n];
       x[i_] := x[i] = be[i] - al[i]*x[i + 1];
```

```
      Table[x[i], {i, 1, n}]
      ];
   oneIter[v_] := Module[{ fv, fu},
       fv = h^2Table[v[[i]], {i, n}];
       u1 = solveTri[fv];
       fu = h^2Table[g[i h, u1[[i]]], {i, 1, n}];
       v1 = solveTri[fu]
       ];
   N[Nest[oneIter, v0, 10]];
   u1
   ]
```

Example 3 *Solve the beam equation*

$$\frac{d^4u}{dx^4} = Exp[-u], \qquad 0 \leq x \leq 1, \tag{39}$$

with the corresponding homogeneous boundary conditions

$$\begin{aligned} u(0) = 0, \quad u''(0) = 0, \\ u(1) = 0, \quad u''(1) = 0, \end{aligned} \tag{40}$$

Solution. We enter data

```
g[x_,u_]:=Exp[-u];
n=9; L=1;
v0=Table[0,{i,n}];
```

and call the module

```
t1=solveFreeEndsImplicit[v0,g,L,n]
```

Let us repeat calculations for $n = 19$

Execute the commands

```
g[x_,u_]:=Exp[-u];
n=19; L=1;
v0=Table[0,{i,n}];
t1=solvFreeEndsImplicit[v0,g,L,n]
```

We give the results in the table rounded off to four digits after decimal point.

x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$u^0_{2h}(x)$	0.0041	0.0077	0.011	0.0124	0.0130	0.0124	0.011	0.0077	0.0041
$u_h(x)$	0.00416	0.0077	0.0105	0.0124	0.0129	0.01230	0.01050	0.0077	0.0041

Table 2

In order to estimate the global error, we apply the Runge's Principle

$$globalError = \frac{1}{3} \max_{1 \le i \le n} |u_{2h}(ih) - u_h)(ih)| = \frac{1}{3} \max_{1 \le i \le n} |t2[i] - t1[i]| = 0.000025$$

The relative error

$$relativeError = 100 \frac{globalError}{\max_{1 \le i \le n} |t2[i]|} = 0.195\%$$

Then, the most correct result

$$\overline{u}_h = \frac{1}{3}(4u_h - u_{2h}).$$

Residual error. For convergent methods the global error is proportional to the residual error. The residual error can be approximately determined by the numerical solution, when the exact solution is unknown. Then, to estimate the residual error, firstly, we find the interpolating polynomial spanned on the list of pairs determined by the table 2 and using the *Mathematica* command

```
pairlist=N[Table[{i h,t2[[i]]},{i,n}]
          {{0.05, 0.002057}, {0.1, 0.00406}, {0.15, 0.005943},
           {0.2,  0.007672}, {0.25,0.009203},{0.3,  0.010502},
           {0.35, 0.011541}, {0.4, 0.012299},{0.45, 0.012759},
           {0.5,  0.012914}, {0.55,0.0127594},{0.6,0.012299},
           {0.65, 0.011541}, {0.7, 0.010502}, {0.75,0.009203},
           {0.8,  0.007672}, {0.85,0.005943}, {0.9, 0.004056},
           {0.95, 0.002057}}
```

Then, we find the interpolating polynomial by execution of the command

```
Clear[x];
pol19 = Simplify[InterpolatingPolynomial[pairlist, x]]
```

$$0.041353\, x - 0.000104\, x^2 - 0.082645\, x^3 + 0.041667x^4 - 0.000346\, x^5 + 2.869905\, 10^{-6} x^6 + 0.0001\, x^7 - 0.000035\, x^8 + 0.00002366\, x^9 - 0.000052\, x^{10} + 0.000095\, x^{11} - 0.000133\, x^{12} + 0.000143\, x^{13} - 0.000116\, x^{14} + 0.00006868\, x^{15} - 0.000027836\, x^{16} + 6.9208865\, 10^{-6}\, x^{17} - 7.9540742\, 10^{-7} x^{18}$$

Next, we evaluate the residual error by the formula

$$residualError = \frac{d^4 u}{dx^4} - f(x, u),$$

replacing u by the pol19. Then, we get the residual error by the commands

```
Clear[x]; n = 19;
residual =
  Table[g[x, pol19] /. x -> i h-D[pol19, {x, 4}]  {i, n}]
```

In the example, the maximal residual error, we calculate by the commands

```
blad = Max[Abs[residual]]

0.0000510502
and relative residual error
relativeblad = 100 blad/Max[Abs[t2]] %
0.39531 %
```

Hence, the residual error 0.00005105 is small and proportional to the global error of the method. Therefore, the global error is also small.

Conclusion. In conclusion, we have proposed stable and convergent algorithms implemented in *Mathematica* modelae *solveFreeEnds* and *solveFreeEndsImplicit* for solving the beam equations. The numerical results given in the Table 1 and Table 2 confirm high accuracy of the results with small global and relative errors. For the further analysis of the errors, we represent the numerical results in the form of interpolating polynomials. Using the interpolating polynomials, we can easily compute the residual errors which for stable algorithms are consistent with the global errors of the methods. As a standard principle used in engineering computations the acceptable results are on the level up 5%. In the examples, the relative errors are: $\approx 1.4\%$ in example 2, $\approx 0.195\%$ in example 3, and the residual error in example 3 with $f(x, u) = e^{-u}$ is $\approx 0.4\%$,. By the principle, these results confirm the effectiveness of the proposed algorithms.

References

1. Clough, R.W. and Penzien J., Dynamics of Structures, McGraw-Hill, 1993.
2. Gere, J.M.,Mechanics of Materials,Nelson Thomas Publishing, 2002.
3. Popov, E.P., Mechanics of Materials, Prentice-Hall, 1978.
4. T. Styš, (2001), A Course in Numerical Methods of Differential Equations,
5. K. Styš & T. Styš, (2000), A First Course in Numerical Analysis, National Institute of Development Research and Documentation, University of Botswana, ISBN99912-2-1492.
6. Wolfram, S., (1992), Mathematica a System for Doing Mathematics by Computers, Addison-Wesley Publishing Company.
7. Euler-Bernoulli Beam Equation, http://en.wikipedia/Euler-Bernoulli-beam-equation.
8. Timoshenko Beam Theory, http://en.wikipedia.org/wiki/Timoshenko-beam-theory