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Lower and upper solutions and existence of $W^{1,1}$-solutions of fuzzy differential equations

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Abstract

The paper is concerned by the existence of $W^{1,1}$-solutions of fuzzy diferential equation

$$u' = f(t, u),$$

with $t \in (a, b)$, f satisfies some Carathéodory conditions and $u(a) = u(b)$. Existence of solutions is given by lower and upper-solutions method and Schauder's fixed point theorem. An application is given and, a result of multiplicity is obtained.

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1 Introduction

This work is concerned by the study of existence of $W^{1,1}$-solutions of the fuzzy differential equation

$$u' = f(t, u), \quad t \in (a, b) \tag{1}$$

with some periodic conditions (see section 3), where $f : [a, b] \times \mathbf{R}_F \to \mathbf{R}_F$ is a Carathéodory function, $[a, b] \subset \mathbf{R}$ with $a < b$, and $\mathbf{R}_F$ is the class of normal and convex fuzzy number defined in section 2.
Our approach is based on the method of lower and upper solutions which relies on standard monotonicity result for function whose derivative is one signed. We obtain existence and location of solutions of (1). The method of lower and upper solution is used, in crisp case, in many applications for first, second and higher order problems for ODE (see e.g. [1,2,5,7] for a thorough account) but, only few results are reported in fuzzy case (see e.g.[6] and reference there in). Our results seem to be new in the theory of fuzzy differential equations, and the techniques used are of independent interest .
Our paper is organized as follows: in section 2, we give some preliminaries in fuzzy sets and functions and define some function spaces : L^p_F for $1 \le p \le \infty$ and $W^{1,1}$, and some of their properties. In section 3, we state and prove our main

result which gives the existence of periodic $W^{1,1}$-solutions of (1). In section 4, we give an application in Ambrosetti-Prodi type problems, and obtain a result of multiplicity.

2 Preliminaries

2.1 Fuzzy sets and functions

Let X be a non empty set, a fuzzy subset of X is a mapping $u : X \to [0, 1]$ (see [4]). The r-level set $[u]^r$ of fuzzy set $u(x)$ on X is defined as

$$[u]^r = \{x \in X \mid u(x) \geq r, \ \ \forall r \in (0, 1]\}.$$

The support set $[u]^0$, is the closure in the topology of X of the union of level sets$[u]^r$, $0 < r \leq 1$. It is well-known that $[u]^r$ is compact for each $0 \leq r \leq 1$, and $[u]^{r_1} \subset [u]^{r_2}$ whenever $0 \leq r_2 \leq r_1 \leq 1$. Let us denote by $\mathbf{R}_F$ the class of fuzzy subsets of $\mathbf{R}$ satisfying the following conditions:

(a) u is normal i.e. $\exists x_0 \in \mathbf{R}$ with $u(x_0) = 1$;

(b) u is convex fuzzy i.e. $\forall \lambda \in [0, 1]$ and $x, y \in \mathbf{R}$, $u(\lambda x + (1 - \lambda)y \geq \min\{u(x), u(y)\}$;

(c) u is upper semicontinuous.

For all $u, v \in \mathbf{R}_F$ and $\lambda \in \mathbf{R}$, the sum $u \oplus v$ and the scalar product $\lambda \odot u$ are well defined by $[u \oplus v]^r = [u]^r + [v]^r$ and $[\lambda \odot u]^r = \lambda[u]^r$, as addition of intervals and dilatation of interval.
A metric is defined on $\mathbf{R}_F$ as follows (see [4]): $d : \mathbf{R}_F \times \mathbf{R}_F \to \mathbf{R}_+ \cup \{0\}$, $(u, v) \mapsto d(u, v) = \sup_{0 \leq r \leq 1} \max(\mid u_-^r - v_-^r \mid, \mid u_+^r - v_+^r \mid)$, where $[u]^r = [u_-^r, u_+^r]$ and $[v]^r = [v_-^r, v_+^r]$.
The metric d satisfies the following properties

(1) $d(u \oplus w, v \oplus w) = d(u, v)$, for all $u, v, w \in \mathbf{R}_F$;

(2) $d(k \odot u, k \odot v) = \mid k \mid d(u, v)$ for $k \in \mathbf{R}$ and $u, v \in \mathbf{R}_F$;

(3) $d(u \oplus w, v \oplus e) = d(u, v) + d(w, e)$ for $u, v, w, e \in \mathbf{R}_F$.

The pair $(\mathbf{R}_F, d)$ forms a complete metric space. The following properties are well-known (see[4]):

Theorem 1 *(i) If $\tilde{0} = \chi_0$, then $\tilde{0} \in \mathbf{R}_F$ and is the neutral element with respect to he sum in $\mathbf{R}_F$. In particular $\mathbf{R} \subset \mathbf{R_F}$;*

(ii) with respect to $\tilde{0}$ and the sum in $\mathbf{R}_F$, none of $u \in \mathbf{R}_F \setminus \mathbf{R}$ has its inverse in $\mathbf{R}_F$.

(iii) for all $h, k \in \mathbf{R}$ with $k, h \geq 0$ and for any $u \in \mathbf{R}_F$, $(k+h) \odot u = k \odot u + h \odot u$;

(iv) for any $\lambda \in \mathbf{R}$, *and* $u, v \in \mathbf{R}_F$, $\lambda \odot (u \oplus v) = \lambda \odot u \oplus \lambda \odot v$;

(v) for any $\lambda, \mu \in \mathbf{R}$ *and* $u \in \mathbf{R}_F$, $\lambda \odot (\mu \odot u) = (\lambda\mu) \odot u$.

Definition 1 *A function* $f : (a, b) \subset \mathbf{R} \to \mathbf{R}_F$ *is said to be strongly generalized differentiable at* $x_0 \in (a, b)$, *if there exists an element* $f'(x_0) \in \mathbf{R}_F$ *such that*

- *(i) for all* $h > 0$ *sufficiently small, there exist* $f(x_0 + h) - f(x_0$, $f(x_0) - f(x_0 - h)$ *and limits in the metric* d

$$\lim_{h \searrow 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \searrow 0} \frac{f(x_0 - h) - f(x_0)}{h} = f'(x_0);$$

or

- *(ii) for all* $h > 0$ *sufficiently small, there exist* $f(x_0) - f(x_0 + h)$, $f(x_0 - h) - f(x_0)$ *and the limits*

$$\lim_{h \searrow 0} \frac{f(x_0) - f(x_0 + h)}{-h} = \lim_{h \searrow 0} \frac{f(x_0 - h) - f(x_0)}{-h} = f'(x_0);$$

or

- *(iii) for all* $h > 0$ *sufficiently small, there exist* $f(x_0 + h) - f(x_0)$, $f(x_0 - h) - f(x_0)$ *and the limits*

$$\lim_{h \searrow 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \searrow 0} \frac{f(x_0 - h) - f(x_0)}{-h} = f'(x_0);$$

or

- *iv for all* $h > 0$ *sufficiently small, there exist* $f(x_0 - f(x_0 + h)$, $f(x_0 - f(x_0 - h)$ *and the limits*

$$\lim_{h \searrow 0} \frac{f(x_0) - f(x_0 + h)}{-h} = \lim_{h \searrow 0} \frac{f(x_0 - f(x_0 - h)}{h} = f'(x_0).$$

Let $\bar{C}[0, 1] = \{\varphi : [0, 1 \to \mathbf{R} \mid \varphi$ is bounded on $[0, 1]$, left continuous for any $t \in (0, 1]$, right continuous on 0 and φ has right limits for any $t \in [0, 1)\}$ endowed with the norm $\| \varphi \|= \sup\{| \varphi(t) | : t \in [0, 1]\}$, $\bar{C}[0, 1]$ is a Banach space , and so is $\bar{C}[0, 1] \times \bar{C}[0, 1]$ with the norm $\| (\varphi, \psi) \|= \max(\| \varphi \|, \| \psi \|)$.

Theorem 2 *(see[3]) If we define* $j : \mathbf{R}_F \to \bar{C}[0, 1] \times \bar{C}[0, 1]$ *by* $j(u) = (u_-, u_+)$, *where* $u_+ : [0, 1] \to \mathbf{R}$ *is a non increasing function for all* $r \in (0, 1]$, *and* $u_- : [0, 1] \to \mathbf{R}$ *is increasing for all* $r \in (0, 1]$, $u_\pm(r) = u^r_\pm$, *then* $j(\mathbf{R}_F)$ *is a closed convex cone with vertex* 0 *in* $\bar{C}[0, 1] \times \bar{C}[0, 1]$ *and* j *satisfies,*

(a) $j(\lambda \odot u \oplus \mu \odot v) = \lambda j(u) + \mu j(v)$ *for any* $u, v \in \mathbf{R}_F$ *and* $\lambda, \mu \geq 0$;

(b) $d(u, v) = \| j(u) - j(v) \|$.

If u is strongly differentiable as in Definition 1(i), then $(j(u))' = j(u')$. To deal with the case (ii), we consider another embedding function $\tilde{j} : \mathbf{R}_F \to \bar{C}[0,1] \times \bar{C}[0,1]$ defined by $\tilde{j}(u) = j((-1) \odot u)$. Clearly, $\tilde{j}$ satisfies conditions (a) and(b) of Theorem 2 and $\tilde{j}(\mathbf{R}_F) = j(\mathbf{R}_F)$. If u is differentiable as in Definition 1(ii), then $(j(u))' = -\tilde{j}(u')$. Note that Definition 1(iii) and (iv) cases can happen on discret set of points. It is also well-known (cfr [1]) that if u is simultaneousy (i) and (ii) at a point t_0, then $u'(t_0) \in \mathbf{R}$

The above theorem implies that j and $\tilde{j}$ are embedding $\mathbf{R}_F$ isometrically and isomorphically into $\bar{C}[0,1] \times \bar{C}[0,1]$. Clearly j (or $\tilde{j}$) is continuous as its inverse. Let M be the set of all functions $u : (a, b) \subset \mathbf{R} \to \mathbf{R}_F$ continuous and strongly generalized differentiable as in Definition 1(i) or (ii).

2.2 Functions spaces

Let $f_r(t) = [f(t)]^r$, for $f : I \subset \mathbf{R} \to \mathbf{R}_F$ The integral of f on I is given by (see[6])

$$[\int_I f(t)\, dt]^r = \int_I f_r(t)\, dt.$$

Let us define the $\mathcal{L}_F^p$ space by

$$\mathcal{L}_F^p(I) = \{f \in C(I, \mathbf{R}_F) \mid f_r \in L^p(I), r \in [0,1]\}$$

for $0 \leq p \leq \infty$. We define the seminorms

$$\| f \|_p = (\int_I (f_r(t))^p\, dt)^{\frac{1}{p}},\ 1 \leq p < \infty$$

and

$$\| f \|_\infty = \inf\{C \mid f_r(t) \subset C \cdot B(0,1), a.e.\, t \in I,\ r \in [0,1]\}.$$

To every $f \in \mathcal{L}_F^p(I)$, one can associate using the function j, the element $(f_-(t), f_+(t))$ of $j(\mathbf{R}_F)$such that $f_\pm^r \in L^p(I)$, for all $r \in [0,1]$. Using this isomorphism-isometry and identifying in $L^p(I)$ functions that are equal almost everywhere on I, one can define the space

$$L_F^p(I) = \{(f_-(t), f_+(t)) \mid f_\pm^r \in L^p(I),\ 1 \leq p \leq \infty\}$$

with norm

$$\| (f_-, f_+) \|_p = \max\{(\int_I \mid f_-^r(t) \mid^p\ dt)^{\frac{1}{p}}, (\int_I \mid f_+^r(t) \mid^p\ dt)^{\frac{1}{p}}\}$$

for $1 \leq p < \infty$ and

$$\| (f_-, f_+) \|_\infty = \max\{(\inf(C_1), \inf(C_2) \mid \mid f_-^r(t) \mid < C_1, \mid f_+^r(t) \mid < C_2,\ a.e.\, t \in I,\ r \in [0,1]\}.$$

The semilinear structure is well defined on $\mathcal{L}_F^p(I)$. Indeed, let f and $g \in \mathcal{L}_F^p(I)$, since

$$[f(t) \oplus g(t)]^r = [f(t)]^r + [g(t)]^r = f_r(t) + g_r(t)$$

and

$$(f_r(t) + g_r(t))^p \leq 2^p((f_r(t))^p + (g_r(t))^p),$$

we have $f_r + g_r \in L^p(I)$ so that $f \oplus g \in \mathcal{L}_F^p(I)$. For any $\lambda \in \mathbf{R}$ and $f \in \mathcal{L}_F^p(I)$, we have $\lambda f_r \in L^p(I)$, so that $\lambda \odot f \in \mathcal{L}_F^p(I)$. Using the isomorphism-isometry j, we get $j(f \oplus g) \in L_F^p(I)$ and $j(\lambda \odot f) \in L_F^p(I)$ for all $\lambda \geq 0$ or $\lambda \leq 0$ where $j((-1) \odot f) = \tilde{j}(f)$.

Definition 2 *A function $f : I \times \mathbf{R}_F \to \mathbf{R}_F$ satisfies a L_F^1-Carathéodory condition if $f(t, \cdot)$ is continuous for a.e. $t \in I$, $f(\cdot, u)$ is strongly measurable (see [6]Def. 2.3.1.) for each $u \in \mathbf{R}_F$, and for every $\rho > 0$ there exists a function $h_\rho \in L^1(I)$ such that for a.e.$t \in I$ and for all $u \in \mathbf{R}_F$ with $d(u, \tilde{0}) \leq \rho$, $\max\{\mid f_-^r(t,u) \mid, \mid f_+^r(t,u) \mid\} \leq h_\rho(t)$.*

Denote by $W^{1,1}(I)$ the space of all functions in M whose first derivative satisfies the above L_F^1-Carathéodory condition.

3 Main result

Let $J \subset \mathbf{R}$ an open interval, and consider $w : J \to \mathbf{R}_F$ a (i)-differentiable solution of equation (1), then w is continuous and has an increasing support. Let $t_0 \in J$ and consider $v : [t_0, +\infty[\to \mathbf{R}_F$, a (ii)-differentiable solution of the initial value problem

$$\begin{aligned} v' &= f(t, v(t)), \quad t > t_0 \\ v(t_0) &= w(t_0). \end{aligned}$$

Then $v(t)$ is continuous and has a decreasing support.
Define

$$\begin{aligned} u(t) &= w(t) \quad \text{if } t \leq t_0, \\ &= v(t) \quad \text{if } t \geq t_0, \end{aligned}$$

then $u(t)$ is continuous and simultaneously (i) and (ii) differentiable at t_0 and $u'(t_0)$ is real. Moreover, one can find $a, b \in J$, with $a < t_0 < b$, such that $[u(a)]^0 = [u(b)]^0$. Hence, we can define the condition

$$u(a) = u(b),$$

and consider the boundary value problem

$$\begin{aligned} u' &= f(t, u)\,, \; t \in (a, b) \subset \mathbf{R} \\ u(a) &= u(b), \end{aligned} \tag{2}$$

where $f : [a, b] \times \mathbf{R}_F \to \mathbf{R}_F$ is a L_F^1-Carathéodory function. We shall use the following notation:
Notation: For all $\alpha, \beta \in \mathbf{R}_F$, $\alpha \preceq \beta$ means $\alpha_- \leq \beta_-$ and $\alpha_+ \leq \beta_+$, $\alpha \prec \beta$ means $\alpha_- < \beta_-$ and $\alpha_+ < \beta_+$, $[\alpha, \beta]_\sim = \{u \mid \alpha \preceq u \preceq \beta\}$, and $(\alpha, \beta)_\sim = \{u \mid \alpha \prec u \prec \beta\}$.
We define a $W^{1,1}$-lower-solution and a $W^{1,1}$-upper-solution as follows

Definition 3 *A function $\alpha : [a,b] \to \mathbf{R}_F$, left continuous and bounded is a $W^{1,1}$-lower -solution of (3), if there exists a partition P: $a = t_1 < t_2 < \cdots < t_n = b$ such that*

(i) $\forall k = 1, 2, \cdots, n-1$, $\alpha \in W^{1,1}(t_k, t_{k+1})$ and $\alpha'(t) \preceq f(t, \alpha(t))$ for a.e. $t \in (t_k, t_{k+1})$;

(ii)1 $\forall k = 2, 3, \cdots, n-1$, $\lim_{t \to t_{k+1}^-} \alpha(t) \succ \lim_{t \to t_k^+} \alpha(t)$ and $\lim_{t \to t_{k-1}^+} \alpha(t) \prec \alpha(t_0)$, if $t \in [a, t_0)$; or

(ii)2 $\forall k = 2, 3, \cdots, n-1$, $\lim_{t \to t_k^+} \alpha(t) \succ \lim_{t \to t_{k+1}^-} \alpha(t)$ and $\lim_{t \to t_{k+1}^-} \alpha(t) \prec \alpha(t_0)$, if $t \in (t_0, b]$.

A function $\beta : [a,b] \to \mathbf{R}_F$ left continuous and bounded is a $W^{1,1}$-upper-solution of (3) if there exists a partition Q :$a = s_1 < s_2 < \cdots < s_m = b$ such that

(i) $\forall i = 1, 2, \cdots, m-1$, $\beta \in W^{1,1}(s_i, s_{i+1})$ and $\beta'(t) \succeq f(t, \beta(t))$ for a.e. $t \in (s_i, s_{i+1})$;

(ii)1 $\forall i = 2, 3, \cdots, m-1$, $\lim_{t \to s_{i+1}^-} \beta(t) \prec \lim_{t \to s_i^+} \beta(t)$ and $\lim_{t \to t_{i-1}^+} \beta(t) \succ \beta(t_0)$, if $t \in [a, t_0)$;or

(ii)2 $\forall i = 2, 3, \cdots, m-1$, $\lim_{t \to s_i^+} \beta(t) \prec \lim_{t \to s_{i+1}^-} \beta(t)$ and $\beta(t_0) \prec \lim_{t \to t_{i+1}^-} \beta(t)$, if $t \in (t_0, b]$.

From now one, we shall call $\alpha(t)$ a lower-solution, and $\beta(t)$ an upper-solution. Our main result is the following

Theorem 3 *Assume $\alpha \preceq \beta$ and define the set*

$$E = \{(t,u) \in [a,b] \times \mathbf{R}_F \mid \alpha(t) \preceq u \preceq \beta(t)\}.$$

If f satisfies a L^1_F-Carathéodory condition on E, then the problem (2) has at least one solution $u \in W^{1,1}(a,b)$ such that for all $t \in [a,b]$, $\alpha(t) \preceq u(t) \preceq \beta(t)$.

Proof. Let us consider the modified problem

$$u' \oplus u = f(t, \gamma(t,u)) \oplus \gamma(t,u) \quad (3)$$
$$u(a) = u(b) \quad (4)$$

where $\gamma : [a,b] \times \mathbf{R}_F \to \mathbf{R}_F$ is defined by

$$\begin{aligned} \gamma(t,u) &= \alpha(t) \quad \text{if} \quad u \prec \alpha(t) \\ &= u \quad \text{if} \quad \alpha(t) \preceq u \preceq \beta(t) \\ &= \beta(t) \quad \text{if} \quad u \succ \beta(t) \end{aligned}$$

We first prove that every solution of (3)-(4) belongs to E. Indeed, assume by contradiction that there exists $t_* \in]a,b[$ and $t^* \in]a,b]$ such that for all $t \in [t_*, t^*]$, with $t \neq t_k$, $k = 1, 2, \ldots, n-1$, $\alpha(t) \succ u(t)$. Using the isomorphism-isometry

j, we get $j \circ u(t) < j \circ \alpha(t)$. Hence, $j \circ \alpha(t) - j \circ u(t)$ achieves a strict positive maximum on $[t_*, t^*]$. Let $t_M \in [t_*, t^*]$ be this maximum point. If $t_M \neq t_k$, then $(j \circ \alpha)'(t_M) - (j \circ u)'(t_M) = 0$. If $[t_*, t^*] \subset (a, t_0]$, then for all $t \in [t_*, t^*]$, we have

$$\begin{aligned} (j(u(t)))' - (j(\alpha(t)))' &= j(u'(t)) - j(\alpha'(t)) \\ &\geq j(\alpha(t)) - j(u(t)) + j(f(t, \gamma(t, u))) - j(f(t, \alpha(t))) \\ &= j(\alpha(t)) - j(u(t)) > 0. \end{aligned}$$

If $[t_*, t^*] \subset [t_0, b)$, then for all $t \in [t_*, t^*]$, we have

$$\begin{aligned} (j(\alpha(t)))' - (j(u(t)))' &= -\tilde{j}(\alpha'(t)) + \tilde{j}(u'(t)) \\ &\geq -\tilde{j}(f(t, \alpha(t))) + \tilde{j}(f(t, \gamma(t, u))) + \tilde{j}(\alpha(t)) - \tilde{j}(u(t)) \\ &= \tilde{j}(\alpha(t)) - \tilde{j}(u(t)) > 0. \end{aligned}$$

Hence, there exists $c > 0$ such that $c < \| (j(\alpha(t)))' - (j(u(t)))' \|$, for all $t \in [t_*, t^*]$. Integrating over $[t_*, t^*]$, we get

$$0 < c \mid t^* - t_* \mid \leq \int_{t_*}^{t^*} \| (j \circ \alpha)'(t) - (j \circ u)'(t) \| \ dt.$$

This being true for all $t \in [t_*, t^*]$, in particular, for $t = t_M$, we have

$$0 < c \mid t^* - t_* \mid \leq 0 \cdot (t^* - t_*) = 0,$$

and we get a contradiction. On the other hand,if $t_M = t_k$, $k = 1, 2, \cdots, n-1$, then there exists $\varepsilon > 0$ such that $j \circ \alpha(t) - j \circ u(t)$ is strictly decreasing for all $t \in (t_k, t_k + \varepsilon]$. Refining, if necessary, the partition P into P_ε which is $\mid t_{k+1} - t_k \mid \leq \varepsilon$ for all $k = \{1, 2, \cdots, n-1\}$, and let $l \in \{1, 2, \cdots, n-1\}$ be such that $t_0 \in]t_l, t_{l+1}[$, then for all $s \in (t_k, t_{k+1})$ using $(ii)1, (ii)2$ we have

$$\begin{aligned} 0 > \sum_{k=1}^{n-1} \int_{t_k}^{t_{k+1}} ((j \circ \alpha)'(s) - (j \circ u)'(s))\, ds &= \sum_{k=1}^{l-1} \{ \lim_{s \to t_{k+1}^-} j \circ \alpha(s) - \lim_{s \to t_k^+} j \circ \alpha(s) \} \\ &\quad + j \circ \alpha(t_0) - \lim_{s \to t_l^+} j \circ \alpha(s) + j \circ u(b) - j \circ u(a) \\ &\quad + j \circ \alpha(t_0) - \lim_{s \to t_{l+1}^-} j \circ \alpha(s) \\ &\quad + \sum_{k=l+1}^{n-1} \{ \lim_{s \to t_k^+} (j \circ \alpha(s) - \lim_{s \to t_{k+1}^-} j \circ \alpha(s) > 0 \end{aligned}$$

and we get a contradiction. These two contradictions, prove that $\alpha(t) \preceq u(t)$. Similar arguments are used to prove that $u(t) \preceq \beta(t)$. We now prove the existence of solutions of (3)-(4). Let us consider the Cauchy problem

$$\begin{aligned} u' &= f(t, u) \\ u(a) &= u_0. \end{aligned}$$

Since $u(a) = u_0 = u(b)$, integrating the above differential equation over $]a, t]$, we obtain

$$u(t) = u_0 \oplus \int_a^t f(s, u(s))\, ds, \tag{5}$$

if $t \in]a, t_0]$, or

$$u(t) = u_0 \oplus \int_a^{t_0} f(s, u(s))\, ds - (-1) \odot \int_{t_0}^t f(s, u(s))\, ds \tag{6}$$

if $t \in]t_0, b[$. Using the isomorphism-isometry j to (5)or (6), we get

$$j \circ u(t) = j \circ u_0 + \int_a^t j \circ f(s, u(s))\, ds, \tag{7}$$

or

$$j \circ u(t) = j \circ u_0 + \int_a^{t_0} j \circ f(s, u(s))\, ds - \int_{t_0}^t \tilde{j} \circ f(s, u(s))\, ds. \tag{8}$$

Since $u_0 = u(a) = u(b)$, we have

$$j(u(t)) = j(u(b)) + \int_a^t j(f(s, u(s)))\, ds, \tag{9}$$

or

$$j(u(t)) = j(u(b)) + \int_a^{t_0} j(f(s, u(s))\, ds - \int_{t_0}^t \tilde{j}(f(s, u(s)))\, ds. \tag{10}$$

Define on $j(\mathbf{R}_F)$ the mapping T by

$$\begin{aligned} Tj \circ u(t) &= j \circ u(b) + \int_a^t j \circ f(s, u(s))\, ds, \quad \text{if} \quad t \in]a, t_0[\\ &= j \circ u(b) + \int_a^{t_0} f(s, u(s))\, ds - \int_{t_0}^t \tilde{j} \circ f(s, u(s))\, ds, \ \text{if} \quad t \in]t_0, b[\end{aligned}$$

Thus fixed points of T are solutions of the integral equation (9)or (10). It is easy to prove that T is bounded and completely continuous in $j(E)$. Since $j(E)$ is convex, closed and bounded, we have a retract of the Banach space $\bar{C}[0,1] \times \bar{C}[0,1]$, and the Schauder's fixed point theorem implies that: T has a fixed point in $j(E)$ which is a solution of (7)or (8). Using the isomorphism j, (5)(or (6)) and (7) (or (8)), we obtain a solution of (3)-(4) in E, which is a solution of (2) in E.

4 Application: a multiplicity result

In this section, we shall work with the embedding function j and use similar argument for $\tilde{j}$ if necessary. Consider the following problem

$$\begin{aligned} (P_s) \quad : \quad & u' + f(t, u) = s \\ & u(a) = u(b) \end{aligned}$$

with $f : [a, b] \times \mathbf{R}_F \to \mathbf{R}_F$ is a L^1_F-Carathéodory function, $s \in \mathbf{R}_F$ a parameter, and $j(f(t, u)) = (f_-(t, u), f_+(t, u))$ such that

(H3)(a) $f_+(t, u) \to +\infty$ and $f_-(t, u) > 0$ as $\| j(u) \| \to \infty$ or ;

(H3)(b) $f_-(t, u) \to -\infty$ and $f_+(t, u) < 0$ as $\| j(u) \| \to \infty$.

Theorem 4 *Assume that (H0), (H1),and (H2) hold. If f satisfies (H3)(a) or (H3)(b), then there exists $s_1 \in \mathbf{R}_F$ with $s_1 \succeq ess \inf_{t\in I}\{\min_{u\in\mathbf{R}} f(t, u)\}$, $I = [a, b]$, such that*

(a) if $s \prec s_1$ then (P_s) has no solution;

(b) if $s = s_1$ then (P_s) has at least one solution;

(c) if $s \succ s_1$ then (P_s) has at least two solutions.

Proof. For any $i \geq 1$, let
$S_i = \{s \in \mathbf{R}_F$ such that (P_s) has at least i solutions $\}$
The proof is divided in five steps
(a) $S_1 \neq \emptyset$. Indeed, taking into account that $\mathbf{R} \in \mathbf{R}_F$ and take $s^* \succ ess \sup_{t\in I}\{f(t, 0)\}$ (i.e. $s^*_- > ess \sup_{t\in I}(f^r_-(t, 0))$ and $s^*_+ > ess \sup_{t\in I}(f^r_+(t, 0))$), then (H3) implies that there exists a real $R_{s^*} > 0$ such that $f(t, R_{s^*}) \succ s^*$ for a.e. $t \in I$. Using theorem 3 with $\alpha = 0$ and $\beta = R_{s^*}$ there exists $u \in \mathbf{R}_F$ which is a solution of (P_{s^*}). Thus $s^* \in S_1$.
(b) If $\tilde{s} \in S_1$ and $s \succeq \tilde{s}$, then $s \in S_1$. Indeed, let $\tilde{u}$ be a solution of $(P_{\tilde{s}})$ and define

$$F_s(\tilde{u})(t) = j(\tilde{u}'(t) \oplus f(t, \tilde{u})) - j(s).$$

Since $j(\tilde{u}'(t) \oplus f(t, \tilde{u})) = j(\tilde{s})$, we have $F_s(\tilde{u})(t) = j(\tilde{s}) - j(s) < 0$, for a.e. $t \in I$. By (H3), there exists $R_s > \max_{t\in I} \| j(u(t) \|$ such that for a.e. $t \in I$,$f(t, -R_s) \succ s$ and $f(t, R_s) \succ s$. Hence, for a.e. $t \in I$, $j(F_s(-R_s))(t) > 0$ and $j(F_s(R_s))(t) > 0$. Applying theorem 3 with $\alpha = -R_s$ and $\beta = \tilde{u} \in \mathbf{R}_F$ or $\alpha = \tilde{u} \in \mathbf{R}_F$ and $\beta = R_s$, we see that (P_s) has at least two solutions u_1 and u_2 with $-R_s \prec u_1(t) \prec \tilde{u}(t)$ and $\tilde{u}(t) \prec u_2(t) \prec R_s$.
(c) If $s_1 = \inf S_1$, then $s_1 \succeq \sigma = ess \inf_{t\in I}(\min_{u\in\mathbf{R}_F} f(t, u))$. Indeed, by (H3)(b), $\sigma = ess \inf_{t\in I}(\min_{u\in\mathbf{R}_F} f(t, u))$ exists. If (P_s) has a solution $u(t)$, then it must satisfies

$$s = \frac{1}{b-a}\int_a^b (u'(t) \oplus f(t, u(t)))\, ds \succeq \sigma,$$

so that $s_1 \succeq \sigma$. If $s \succ s_1$, then there exists $\tilde{s} \in (s_1, s)_\sim \cap S_1$, and by the part (b), $s \in S_2$.
(d) For each $\tilde{s} \succ s_1$, the set of all solutions of (P_s) with $s \preceq \tilde{s}$ is a priori bounded. Indeed, let $R_{\tilde{s}} > 0$ a real be such that $f(t, v) \succ \tilde{s}$ whenever $\| j(v) \| \geq R_{\tilde{s}}$, for a.e. $t \in I$ and let $u \in W^{1,1}(I)$ be a possible solution of (P_s). If there exists $\bar{t} \in (a, b)$ such that

$$\max_{t\in I} u(t) = u(\bar{t})$$

then, there exists a sequence $(t_n) \subset [a, b]$ with $t_n \to \bar{t}$ such that $u'(t_n)$ exists in the generalized sense (i) or (ii) for each $n \geq 0$ and

$$u'(t_n) \oplus f(t_n, u(t_n)) = s.$$

Thus

$$j(u'(t_n) + j(f(t, u(t_n))) = j(s).$$

Since $u(t)$ is strongly differentiable in the generalized sense (i) or (ii), $j \circ u$ is Fréchet differentiable and by continuity, we have $(j \circ u)'(\bar{t}) = 0$ that is $j \circ u'(\bar{t}) = 0$ or $-\tilde{j} \circ u'(\bar{t}) = 0$. Hence, $j(\mathbf{R}_F)$ being a cone with vertex 0, (t_n) can be chosen such that $j \circ u'(t_n) \geq 0$ or $-\tilde{j} \circ u'(t_n) \geq 0$. Thus

$$j(s) - j(f(t_n, u(t_n))) = j \circ u'(t_n \geq 0$$

or

$$\tilde{j}(s) - \tilde{j}(f(t_n, u(t_n))) = \tilde{j} \circ u'(t_n) \leq 0$$

and

$$0 \leq j(s) - j(f(t_n, u(t_n))) \leq j(\tilde{s}) - j(f(t_n, u(t_n))).$$

Hence

$$j(f(t_n, u(t_n))) \leq j(\tilde{s})$$

for every $n \in \mathbf{N}$, taking into account that $s \preceq \sigma$ implies that $\tilde{j}(\sigma) \leq \tilde{j}(s)$ we get similar relations for $\tilde{j}$. Hence $f(t_n, u(t_n)) \preceq \tilde{s}$ for all $n \in \mathbf{N}$. Therefore, $\| j(u(t_n)) \| < R_{\tilde{s}}$. Hence $u(t_n)$ converges uniformly. Passing to the limit we have

$$\||| j(u(\bar{t})) \| < R_{\tilde{s}}.$$

If $\bar{t} = a$, then since $j \circ u(a) = j \circ u(b)$, we have

$$\max_{t \in I} j \circ u(t) = j \circ u(a) = j \circ u(b),$$

and hence $j \circ u'(a) \leq 0 \leq j \circ u'(b)$. Therefore, $j(f(a, u(a))) - j(s) \leq 0$ and $j(f(a, u(a))) \leq j(s) \leq j(\tilde{s})$ imply that $\| j(u(a)) \| < R_{\tilde{s}}$, and $0 \leq j \circ u'(b) = j(s) - j(f(b, u(b))) \leq j(\tilde{s}) - j(f(b, u(b)))$, implies that $\| j(u(b)) \| < R_{\tilde{s}}$.
(e) $s_1 \in S_1$. Indeed, let (σ_n) be a decreasing sequence in

$$T_1 = \{s \in \underline{\mathrm{R}}_F \mid s \succeq s_1\}$$

such that $\sigma_n \to s_1$, and for all $n \in \mathbf{N}$, let $(u_n)_{n \in \mathbf{N}}$ be the corresponding sequence of solutions of (P_s). If R_{σ_1} is the a priori bound of the part (d), then

$$\max_{t \in I} \| j(u_n(t)) \| < R_{\sigma_1}.$$

Since,

$$u_n'(t) \oplus f(t, u_n(t)) = \sigma_n,$$

we have

$$j \circ u_n'(t) + j(f(t, u_n(t)) = j(\sigma_n),$$

and integrating for $t \in (a,b)$, we get

$$j \circ u_n(t) - j \circ u_n(a) = j(\sigma_n)(t-a) - \int_a^t j(f(s,u_n(s)))\,ds$$

or

$$j \circ u_n(a) - j \circ u_n(t) = \tilde{j}(\sigma_n)(t-a) - \int_a^t \tilde{j}(f(s,u_n(s)))\,ds$$

thus

$$j \circ u_n(t) - j \circ u_n(b) = j(\sigma_n)(t-a) - \int_a^t j(f(s,u_n(s)))\,ds; \tag{11}$$

or

$$j \circ u_n(b) - j \circ u_n(t) = \tilde{j}(\sigma_n)(t-a) - \int_a^t \tilde{j}(f(s,u_n(s)))\,ds, \tag{12}$$

and without loss of generality, let us fixe $u_n(b) = u_0$ for all $n \in \mathbf{N}$, and take into account that $\| j(f) - j(g) \| = \| \tilde{j}(f) - \tilde{j}(g) \|$, for all $m, n \geq 0$, we have

$$\begin{aligned} \| j(u_m)(t) - j(u_n)(t) \| &= \| \int_a^t j(f(s,u_n(s))) - j(f(s,u_m(s)))\,ds + t-a)(j(\sigma_m) - j(\sigma_n)) \| \\ &\leq \int_a^t \| j(f(s,u_n(s))) - j(f(s,u_n(s))) \| \; ds \\ &\quad + \int_a^t \| j(\sigma_m) - j(\sigma_n) \| \; ds \\ &\leq \int_a^t \| j(f(s,u_n(s))) - j(f(s,u_m(s))) \| \; ds \\ &\quad + \int_a^t \{\| j(\sigma_m) - j(s_1) \| + \| j(s_1) - j(\sigma_n) \|\}\,ds \\ &\leq \frac{\epsilon}{3}(b-a) + \frac{\epsilon}{3}(b-a) + \frac{\epsilon}{3}(b-a) = \epsilon(b-a). \end{aligned}$$

Hence $j \circ u_n(t))$ is a Cauchy sequence in $C([a,b], j(\mathbf{R}_F))$ with norm $\| z \|_0 = \max_{t \in I} \| z(t) \|$. By the completeness, we have $j(u_n(t)) \to j(u(t))$.
Define $g_n(t) \in \mathbf{R}_F$ by

$$g_n(t) \oplus f(t,u(t)) = f(t,u_n(t))$$

then,

$$j \circ g_n(t) = j \circ f(t,u_n(t)) - j \circ f(t,u(t)).$$

Since $j \circ f(t,\cdot)$ is continuous for a.e. $t \in I$, we have

$$\begin{aligned} \int_a^t \| j \circ g_n(s) \| \; ds &= \int_a^t \| j \circ f(s,u_n(s)) - j \circ f(s,u(s)) \| \; ds \\ &\leq \epsilon(b-a), \end{aligned} \tag{13}$$

for all $\epsilon > 0$. Hence, as $n \to \infty$, $j \circ g_n(t) \to 0$ for a.e. $t \in I$. Since f is a L^1_F-Carathéodory function, we have

$$\int_a^t \| j \circ f(s, u(s)) \| \; ds < \infty$$

for all u such that $\| j \circ u(t) \| \leq \rho$ with $\rho \geq R_{\sigma_1}$. Denote by $\varphi_\epsilon(g_n(t)) = \| j \circ g_n(t) \|$ and $\psi_\epsilon(f(t, u(t))) = \| j \circ f(t, u(t)) \|$. Then

$$\| j(f(t, u(t)) \oplus g_n(t)) - j(g_n(t)) \| \leq \epsilon \varphi_\epsilon(g_n(t) \oplus \psi_\epsilon(f(t, u(t))).$$

Since we have,

(i) $j \circ g_n(t) \to 0$ a.e.$t \in I$;

(ii) $j \circ f \in L^1$;

(iii) $\int_a^b \varphi_\epsilon(g_n(t))\, dt \leq C < \infty$;

(iv) $\int_a^b \psi_\epsilon(f(t, u(t)))\, dt < \infty$,

using the Brezis-Lieb Lemma (see[4] Theorem 2), we obtain

$$\int_a^t \| j(g_n(s) \oplus f(s, u(s))) - j(g_n(s)) - j(f(s, u(s))) \| \; ds \to 0$$

as $n \to \infty$ for all $t \in I$. Thus

$$\lim_{n\to\infty} \int_a^t j(g_n(s))\, ds = \lim_{n\to\infty} \int_a^t \{j(f(s, u_n(s))) - j(f(s, u(s)))\}\, ds.$$

Using the dominated convergence and (13), we have

$$\lim_{n\to\infty} \int_a^t j(f(s, u_n(s)))\, ds = \int_a^t j(f(s, u(s)))\, ds.$$

Hence, passing to the limit in (11) or (12), we get

$$j \circ u(t) - j \circ u(b) = j(s_1)(t - a) - \int_a^t j \circ f(s, u(s))\, ds$$

or

$$j \circ u(b) - j \circ u(t) = \tilde{j}(s_1)(t - a) - \int_a^t \tilde{j} \circ f(s, u(s))\, ds$$

for all $t \in I$. Since $j \circ u(a) = j \circ u(b)$, u satisfies (P_{s_1}). Thus, $s_1 \in S_1$.

Remarks. The theory in this paper, though it gives similar results as in classical case, is more rich than in classical case, since the behaviour of functions in fuzzy case is completely different than the behaviour in classical case. Indeed

(1) Fuzzy solutions of (1) are mappings assigning a membership grade taking value in $[0, 1]$ to elements of a nonempty base set.

(2) Some concepts such as periodicity or stability cannot be defined as in classical case, since in fuzzy case solutions of the initial value problem suffer the disadventage of having increasing support. For the periodic solutions, we overcame this shortcoming by using a pasting approach.

(3) The $W^{1,1}$-lower and upper solutions defined in definition 3, have a behaviour completely different than in classical case.

(4) When a real world problem or phenomenon is transformed by a deterministic model into a boundary or initial value problem of equation (1), it is not usually sure that the model is perfect. It could contain some incertainty, for example the function f could contain uncertain parameters or the boundary values could not be known exactly. The estimation of solutions will necessarly be subject to some errors. If the nature of this errors is random, stochastictic differential equations with random datas could be used, but if the underlying structure is not probabilistic, the best way to manage such a problem is to use fuzzy differential equations. Therefore, the theory used in the paper is more general and more rich than in the classical case.We point out that $\mathbf{R} \subset \mathbf{R_F}$.

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