Volume 4 (2009) 1 - 12

Southern Africa Journal of Pure and Applied Mathematics

SAJPAM
ISSN:1993-8594
www.samsajournal.org

Limit Cycle and existence of periodic solutions of small pertubations of autonomous systems of fuzzy differential equations

Rebecca Walo Omana,
Department of Mathematics and Computer Sciences,
University of Kinshasa, B.P. 190 Kinshasa 11, D.R. Congo

Abstract

The paper deals with the existence of periodic solution of periodically perturbed autonomous systems of fuzzy differential equations when the autonomous system has a limit cycle. Our approach is based on fixed point index of compact mappings.

Keywords: Fuzzy differential equations; Periodic solutions; Fixed point index; limit cycle.
AMS Subject classification 34A34, 47H14, 34A99.

1 Introduction

In the crisp case, the Nemitskii's conjecture, deals with the existence of ω-periodic solutions of periodic pertubations of autonomous systems of first order differential equations of the form

$$\begin{aligned} x' &= X(x,y) \\ y' &= Y(x,y), \end{aligned} \tag{1}$$

in the presence of a limit cycle of this system. This conjecture was first settled by A. Halanay [7] as follows: if the autonomous system (1) has a limit cycle, then there is at least one ω-periodic solution of the perturbed system

$$\begin{aligned} x' &= X(x,y) + \varepsilon E_1(t) \\ y' &= Y(x,y) + \varepsilon E_2(t) \end{aligned} \tag{2}$$

Many papers dealing with the systems (1)and (2) have been published see for instance J. Cronin [3,4,5] and A.C. Lando [10,11]. More significant and easy proof of this conjecture is given in ([2], corollary 5), see also [8].
In this paper, we deal with the Nemitskii's conjecture in fuzzy case for the system of equations

$$u' = F(t,u;\beta) \tag{3}$$

with $F(t,u;\beta) = f(u) + \beta\psi(t,u)$, where $f : \mathbf{R}_F^2 \to \mathbf{R}_F^2$ is Lipschitz continuous, $\psi : [0,T] \times \mathbf{R}_F^2 \to \mathbf{R}_F^2$ is T_1-periodic and continuous, $\mathbf{R}_F^2 = \mathbf{R}_F \times \mathbf{R}_F$ and $\mathbf{R}_F$ is the set of normal convex fuzzy-numbers. Our approach is simple. We embed the set $\mathbf{R}_F^2$ into a Banach space as a closed convex cone and construct an absolute retract of this space. Using the index theory in retract and the fixed point index of compact mappings, we show the existence of periodic solutions in a bounded open set whose boundary is close to the limit cycle. The paper is organized as follows: In section 2, we give some preliminaries in fuzzy-number valued functions. In section 3 we state and show our main result.

2 Preliminaries

Denote $\mathbf{R}_F$ the set of fuzzy numbers $u : \mathbf{R} \to [0,1]$ such that

(a) u is normal i.e. there exists $x_0 \in \mathbf{R}$ such that $u(x_0) = 1$;

(b) u is fuzzy convex i.e; for all $\lambda \in [0,1]$, and $x, y \in \mathbf{R}$, $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$;

(c) u is upper semicontinuous, and $[u]^0 = Cl\{x \in \mathbf{R} \mid u(x) > 0\}$ is compact.

For $0 < \alpha \leq 1$, denote $[u]^\alpha = \{x \in \mathbf{R} \mid u(x) > \alpha\}$ the α-level set for u. Define the functions $u_-, u_+ : [0,1] \to \mathbf{R}$ such that $u_- : [0,1] \to \mathbf{R}$ is non decreasing and continuous, $u_+ : [0,1] \to \mathbf{R}$ is non increasing and continuous. Denote $u_\pm(\alpha) = u_\pm^\alpha$, then one can write $[u]^\alpha = [u_-^\alpha, u_+^\alpha]$ and $[u]^\alpha$ is a real interval.
Define $d : \mathbf{R}_F \times \mathbf{R}_F \to \mathbf{R}^+ \cup \{0\}$ by the equation

$$d(u,v) = \sup_{0\leq\alpha\leq 1} (\max \mid u_-^\alpha - v_-^\alpha \mid, \mid u_+^\alpha - v_+^\alpha \mid).$$

It is easy to show that d is a metric in $\mathbf{R}_F$, moreover the following is well-kown (see [1]):

(i) $(\mathbf{R}_F, d)$ is a complete metric space;

(ii) $d(u \oplus w, v \oplus w) = d(u,v)$for all $u, v, w \in \mathbf{R}_F$;

(iii) $d(\lambda \odot u, \lambda \odot v) =\mid \lambda \mid d(u,v)$ for all $u, v \in \mathbf{R}_F$ and $\lambda \in \mathbf{R}$

Define the product set $\mathbf{R}_F^2 = \mathbf{R}_F \times \mathbf{R}_F$, and $d_0 : \mathbf{R}_F^2 \times \mathbf{R}_F^2 \to \mathbf{R}^+ \cup \{0\}$ by the equation

$$d_0(u,v) = d(u_1,v_1) + d(u_2,v_2).$$

Clearly $(\mathbf{R}_F^2, d_0)$ is a complete metric space. Denote $\overline{C}[0,1] = \{\varphi : [0,1] \to \mathbf{R} \mid \varphi$ is bounded on $[0,1]$, left continuous on $(0,1]$, right continuous at 0 and has right limit for any $t \in [0,1)\}$. Then $\overline{C}[0,1]$ is a Banach space with the norm $\parallel \varphi \parallel = \sup\{\mid \varphi(t) \mid : t \in [0,1]\}$, so is the product space $\overline{C}[0,1] \times \overline{C}[0,1]$ with the norm $\parallel (\varphi, \psi) \parallel = \max\{\parallel \varphi \parallel, \parallel \psi \parallel\}$. Denote $\mathcal{E} = \overline{C}[0,1]^2$. The metric space $(\mathbf{R}_F, d)$ can be embedded into $\mathcal{E}$ isometrically-isomorphicaly by the following embedding theorem

Theorem 1 *(see [1]). If we define the function $j : \mathbf{R}_F \to \mathcal{E}$ by $j(u) = (u_-, u_+)$ where u_+, u_- are the functions discussed above, then j is invertible and continuous as its inverse, and embeds $\mathbf{R}_F$ into $\mathcal{E}$ isomorphically and isometrically as a closed convex cone with vertex 0. Moreover j satisfies the following equations*

(a) $\| j(u) - j(v) \| = d(u, v)$ fo all $u, v \in \mathbf{R}_F$;

(b) $j(\lambda \odot u \oplus \mu \odot v) = \lambda j(u) + \mu j(v)$ for all $u, v \in \mathbf{R}_F$ and $\lambda, \mu \geq 0$.

For the multiplication with a negative number, we have the following embedding function (see [1] Remark 6): $\tilde{j} : \mathbf{R}_F \to \mathcal{E}$, $\tilde{j}(u) = j((-1) \odot u)$, $u \in \mathbf{R}_F$. This embedding function satisfies the following properties: $\| \tilde{j}(u) - \tilde{j}(v) \| = d(u, v)$, $j(\mathbf{R}_F) = \tilde{j}(\mathbf{R}_F)$ and $\tilde{j}(\lambda \odot u \oplus \mu \odot v) = \lambda \tilde{j}(u) + \mu \tilde{j}(v)$, for all $u, v \in \mathbf{R}_F$, $\lambda, \mu \geq 0$. Denote $j_*(u) = (j(u_1), j(u_2))$ for all $u = (u_1, u_2) \in \mathbf{R}_F^2$,and $\mathcal{E}^2 = \mathcal{E} \times \mathcal{E}$, then $\mathcal{E}^2$ is a Banach space with the norm

$$\| (\Phi, \Psi) \| = \| \Phi \| + \| \Psi \|,$$

and j_*embeds $\mathbf{R}_F^2$ into $\mathcal{E}^2$. Moreover

$$\begin{aligned} \| j_*(u) - j_*(v) \| &= \| (j(u_1) - j(v_1), j(u_2) - j(v_2)) \| \\ &= \| j(u_1) - j(v_1) \| + \| j(u_2) - j(v_2) \| \\ &= d(u_1, v_1) + d(u_2, v_2) = d_0(u, v). \end{aligned} \tag{4}$$

Clearly $j_*(\lambda \odot u \oplus \mu \odot v) = \lambda j_*(u) + \mu j_*(v)$ for all $u, v \in \mathbf{R}_F^2$ and $\lambda, \mu \geq 0$. Therefore, $\mathbf{R}_F^2$ is embedded isometrically and isomorphically into $\mathcal{E}^2$, as a closed convex cone with vertex $0 = (0, 0)$.
We have the following definition of differentiability from the reference [1].

Definition 1 *A function $f : (a, b) \subset \mathbf{R} \to \mathbf{R}_F$ is said to be strongly generalized differentiable at $x_0 \in (a, b)$, if there exists an element $f'(x_0) \in \mathbf{R}_F$ such that*

- *(i) for all $h > 0$ sufficiently small, there exist $f(x_0 + h) - f(x_0$, $f(x_0) - f(x_0 - h)$ and limits in the metric d*

$$\lim_{h \searrow 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \searrow 0} \frac{f(x_0) - f(x_0 - h)}{h} = f'(x_0);$$

or

- *(ii) for all $h > 0$ sufficiently small, there exist $f(x_0) - f(x_0 + h)$, $f(x_0 - h) - f(x_0)$ and the limits*

$$\lim_{h \searrow 0} \frac{f(x_0) - f(x_0 + h)}{-h} = \lim_{h \searrow 0} \frac{f(x_0 - h) - f(x_0)}{-h} = f'(x_0);$$

or

- *(iii) for all $h > 0$ sufficiently small, there exist $f(x_0 + h) - f(x_0)$, $f(x_0 - h) - f(x_0)$ and limits*

$$\lim_{h \searrow 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \searrow 0} \frac{f(x_0 - h) - f(x_0)}{-h} = f'(x_0);$$

or

- *iv for all $h > 0$ sufficiently small, there exist $f(x_0 - f(x_0 + h)$, $f(x_0 - f(x_0 - h)$ and the limits*

$$\lim_{h \searrow 0} \frac{f(x_0) - f(x_0 + h)}{-h} = \lim_{h \searrow 0} \frac{f(x_0 - f(x_0 - h)}{h} = f'(x_0).$$

Remark 1. (1) If $u : (a, b) \to \mathbf{R}$ is strongly differentiable on $x_0 \in (a, b)$ according to (i), then $(j_*(u))' = j_*(u')$. If the case (ii) holds, then using the embedding $\tilde{j} : \mathbf{R}_F \to \mathcal{E}$, we have $(j(u))' = -\tilde{j}(u') = -j((-1) \odot u')$. If the case$(iii)$ or (iv) holds then $j \circ u$ is Fréchet differentiable at right and at left of x_0 and $(j(u))'_R(x_0) = j(u'(x_0))$, $(j \circ u)'_L(x_0 = -\tilde{j}(u'(x_0))$ or $j \circ u$ is Fréchet differentiable at left and at right of x_0 and $(j \circ u)'_L(x_0) = j(u'(x_0)$ and $(j \circ u)'_R(x_0) = -\tilde{j}(u'(x_0))$. The cases (iii) and (iv) can happen on a discrete set of points.
(2) According to the part (1), for $(u, v) \in \mathbf{R}^2_F$, using the embedding function j_*, we have either $(j_*(u, v))' = j_*(u', v')$ or $(j_*(u, v))' = (-\tilde{j}(u'), -\tilde{j}(v') = -\tilde{j}_*(u', v')$, or $(j_*(u, v))' = (j(u'), -\tilde{j}(v'))$ or $(j_*(u, v))' = (-\tilde{j}(u'), j(v'))$ if u and v are strongly differentiable according to the cases (i), or (ii) or (iii) or (iv).
(3) If u is simultaneously strongly differentiable in the generalized sense (i) and (ii) at a point $x_0 \in (a, b)$, then $u'(x_0)$ is real.
(4) If the cases (i) or (ii) hold, then for $w \in \mathbf{R}^2_F$, we have

$$\int_0^t j_*(w')(s)\, ds = j_*(w(t)) - j_*(w(0))$$

or

$$\int_0^t \tilde{j}_*(w')(s)\, ds = j_*(w(0)) - j_*(w(t)).$$

Remark 2. Let $J \subset \mathbf{R}$ and $w : J \to \mathbf{R}_F$ be a strongly differentiable function in the generalized sense (i) satisfying equation (3), and let $x_0 \in J$ and $v(x)$ a solution of equation (3) (ii)-differentiable in the generalized sense, with initial condition $v_0 = v(x_0) = w(x_0)$, and define the function $u(x)$ by

$$\begin{aligned} u(x) &= w(x), \ \ x \leq x_0 \\ &= v(x), \ \ x \geq x_0 \end{aligned}$$

Clearly $u(x)$ is continuous, satisfies the equation (3), and has increasing support on the interval $(x_0 - \delta, x_0]$, and decreasing support on the interval $[x_0, x_0 + \delta)$

for some $\delta > 0$ real. It follows that u is simultaneously (i) and (ii)- strongly differentiable in the generalized sense at x_0. Therefore, $u'(x_0)$ is real, moreover, one can find $a < b$ on $(x_0 - \delta, x_0 + \delta)$ such that u has the same support at a and at b, and define the condition:

$$u(a) = u(b) \tag{5}$$

Using translation if necessary, one can define the condition

$$u(0) = u(T) \tag{6}$$

for some $T > 0$.

Let us define the set $M = \{u : [0, T] \to \mathbf{R}_F^2 \mid u(t)$ piecewise continuously differentiable, and defined as in Remark 2 $\}$, and define the distance in M by the equation

$$D_0(u, v) = \sup_{t \in [0,T]} d_0(u(t), v(t)),$$

then, clearly (M, D_0) is a complete metric space. Denote $j_*(M) = \{j_*(u) \mid u \in M\}$, and

$$D_0^*(j_*(u), j_*(v)) = \sup_{t \in [0,T]} \| j_*(u)(t) - j_*(v)(t) \| .$$

Then, $(j_*(M), D_0^*)$ is a complete metric space. Since $j_*(\mathbf{R}_F^2)$ is an absolute retract of the Banach space $\mathcal{E}^2$, then so is $j_*(M)$.

3 The main result

3.1 Existence of a limit cycle in $\mathbf{R}_F^2$

Consider the autonomous system of fuzzy differential equations

$$u' = f(u) \tag{7}$$

where $u(.) \in M$. Using the embedding functions j_* and $\tilde{j}_*$, by Remark 1 we have the system

$$j_*(u') = j_*(f(u)), \tag{8}$$

or

$$\tilde{j}_*(u') = \tilde{j}_*(f(u)). \tag{9}$$

Rescaling the time interval by $t = T \cdot \tau$ where $T = T(T_0, T_1)$, $\tau > 0$ and $\frac{T_1}{T_0} = k \in \mathbf{Q}$, and denote $(u(t))' = \frac{d}{dt} u(t)$, we obtain

$$j_*(u') = Tj_*(\frac{d}{d\tau} u) = Tj_*(u') = Tj_*(f(u)) = j_*(T \odot f(u)), \tag{10}$$

or

$$\tilde{j}_*(u') = T\tilde{j}_*(u') = T\tilde{j}_*(f(u)) = \tilde{j}_*(T \odot f(u)). \tag{11}$$

Hence,

$$u' = T \odot f(u) \tag{12}$$

and $0 \leq t \leq T$ if only if $0 \leq \tau \leq 1$. Let $y(t) = u(\frac{t}{T})$, then y is T-periodic solution of (12) if and only if u is a 1-periodic solution of (12). Let τ_0 be the pasting point x_0 of Remark 2, then integrating (12) we get

$$u(\tau) = u(1) \oplus T \odot \int_0^\tau f(u(s))\, ds, \tag{13}$$

if $0 < \tau < \tau_0$ or

$$u(\tau) = u(1) \oplus (T) \odot \int_0^{\tau_0} f(u(s))\, ds - (-T) \odot \int_{\tau_0}^{\tau} f(u(s))\, ds, \tag{14}$$

if $\tau_0 < \tau < 1$. Thus,

$$\begin{aligned} j_*(u(\tau)) &= j_*(u(1) \oplus T \int_0^\tau f(u(s))\, ds) \\ &= j_*(u(1)) + T \int_0^\tau j_*(f(u(s)))\, ds, \end{aligned} \tag{15}$$

or

$$j_*(u(\tau)) = j_*(u(1)) + T \int_0^{\tau_0} j_* f(u(s))\, ds - T \int_{\tau_0}^{\tau} \tilde{j}_*(f(u(s)))\, ds. \tag{16}$$

The right hand part of (15) or (16) defines a compact operator Q_T on the space of continuous functions on $[0, 1]$, with values in $\mathcal{E}^2$ such that:

$$(Q_T j_*(u))(\tau) = j_*(u(1)) + T \int_0^\tau j_*(f(u(s)))\, ds. \tag{17}$$

if $0 < \tau < \tau_0$ or

$$(Q_T j_*(u))(\tau) = j_*(u(1)) + T \int_0^{\tau_0} j_*(f(u(s)))\, ds - T \int_{\tau_0}^{\tau} \tilde{j}_*(f(u(s)))\, ds. \tag{18}$$

for $\tau_0 < \tau < 1$. If $v_0(\tau) = j_*(u(\tau))$ is a solution of (15)or (16) for $T = T_0$, then so is $v_0(\tau + h)$ for $h > 0$ small enough. Therefore, $v_0(\tau)$ forms a curve C_0 in $C^1([0, 1], \mathcal{E}^2)$. Assume the following condition:
(A_0) For all $T > 0$ such that $0 < T < T_0$ and $T_0 < T < 2T_0$, Q_T has no fixed point.
Then, C_0 is an isolated curve. Moreover $u(\tau)$, 1-periodic implies that C_0 is a closed curve in $\mathcal{E}^2$. By the definition and properties of j_*, there exists an isolated periodic solution in M, whose trajectory corresponds to C_0. Therefore, we may assume the following condition:
(A_1) The autonomous system (7) has a limit cycle $u_0 = u_0(t)$ of least period T_0 whose trajectory corresponds to a closed curve C_0 in $j_*(M)$.
Following [2], and [8],and using the properties of j_*, we construct a family of

bounded open sets in M, neighbourhoods of the limit cycle, and whose boundary is closed to the limit cycle. Let O_0 be the interior of the curve C_0, and consider the family of bounded open neighbourhoods of C_0 in $\mathcal{E}^2$ defined by

$$\Omega(O_0, \gamma) = O_0 \setminus (C_0 + \mid \gamma \mid B(0,1))$$

if $\gamma < 0$, and

$$\Omega(O_0, \gamma) = O_0 \cup (C_0 + \mid \gamma \mid B(0,1))$$

if $\gamma > O$. Hence,

$$B(C_0, \gamma) = [(\Omega(O_0, \gamma) \cup O_0) \setminus O_0]_{\gamma > 0} \cup [(\Omega(O_0, \gamma) \cup O_0) \setminus \Omega(O_0, \gamma)]_{\gamma < 0} \to C_0$$

as $\mid \gamma \mid \to 0$. Clearly $B(C_0, \gamma)$ is a bounded open set and for $v \in \partial B(C_0, \gamma)$, we have either $v \in \partial\Omega(O_0, \gamma)$ or $v \in C_0$.

3.2 Existence theorems

Consider the Cauchy problem

$$\begin{aligned} u' &= F(t, u; \beta) \\ u(0) &= \xi, \end{aligned} \tag{19}$$

and apply the isometry-isomorphism j_* to (19), we get another Cauchy problem

$$\begin{aligned} j_*(u') &= j_*(F(, u; \beta)) \\ j_*(u(0)) &= j_*(\xi). \end{aligned} \tag{20}$$

Nagumo's theorem (see [2] and reference therein) ensures that (20) has a solution $v(\cdot) : dom\, v(\cdot) \to j_*(\mathbf{R}_F^2)$ defined on a right maximal neighbourhood of 0 if and only if
(A_2) $j_*(F(t, u, \beta)) \in \mathcal{T}(j_*(u), j_*(\mathbf{R}_F^2))$ for all $t \in [0, T]$, $T = T(T_0, T_1)$, $u \in j_*^{-1}(\partial j_*(\mathbf{R}_F^2))$, where $\mathcal{T}(v, j_*\mathbf{R}_F^2)$ is the Bouligand cone tangent to $j_*(\mathbf{R}_F^2)$ at v.

Based on the previous section we can obtain similar results if we apply the isometry-isomorphism $\tilde{j}_*$.
Let $u(t) = \omega(t, 0, \xi)$ be the unique and maximal solution of

$$\begin{aligned} u' &= F(t, u; 0) \\ u(0) &= \xi. \end{aligned} \tag{21}$$

We assume the following condition:
(A_3) $\omega(t, 0, \xi)$ is Fréchet differentiable and $\omega_3'(t, 0, \xi)$ is continuously invertible, where $\omega_k(x_1, x_2, x_3) = \frac{\partial}{\partial x_k}\omega(x_1, x_2, x_3)$, $k = 1, 2, 3$.
Our main existence result can be formulated as follows

Theorem 2 . *Assume* (A_1),(A_2),(A_3) *hold and*

(A_4) *there is no periodic solution of (3) on* $j_*^{-1}(\partial\Omega_0)$, *where* $\Omega_0 = \Omega(C_0, \gamma)$ $\gamma \in (-\gamma_0, \gamma_0)$ *for all* $\beta > 0$.

(A_5) $\chi(C_0) \neq 0$,

where $\chi(\Gamma)$ *is the index of the cycle* Γ. *Then, there exists* $\beta_0 > 0$ *small such that for every* $\beta \in (0, \beta_0)$, *(3) has a* T*-periodic solution* u_β *lying in the set* $S_0 = \{u : [0,T] \to M \mid u(t) = \omega(t, 0, \xi)$, *with* $j_*(\xi) \in B(C_0, \gamma)$ *and* $u(t) \to u_0(t)$ *as* $\beta \to 0\}$.

The following Lemma is well-known (see[9])

Lemma 1 . *The index of a cycle which is orbitally asymptotically stable is equal to* $+1$.

In particular, (A_5) is satisfied if C_0 is orbitally asymptotically stable. This can be achieved if C_0 is a non degenerate cycle i.e. if 1 is not a Floquet multiplier of the linearized system of (7). To deal with this case, we suppose the following condition
$(A_5)'$ the linear system

$$y' = (j_* \circ f)'(u_0(t))y \tag{22}$$

doesnot have a $2T_0$-periodic solution linearly independent to $j_*(u_0)$.
Clearly, $(A_5)'$ contains (A_0) and (A_5). Indeed, by $(A_5)'$, C_0 is an isolated cycle, and the multipliers are different from -1 and 1, which means that C_0 is either asymptotically stable or unstable. therefore, in a sufficiently small neighbourhood of C_0, the only fixed points of Q_T are those belonging to C_0. Thus, if $\gamma_0 > 0$ is small enough then, the only fixed points of Q_T in S_0 are those belonging to C_0. By the Lemma 1, $\chi(C_0) = 1$, and we have:

Corollary 1 . *Assume* (A_1), (A_2), (A_3), (A_4) *and* $(A_5)'$ *hold. Then, (3) has at least a* T*-periodic solution lying in the region* S_0.

3.3 Proof of the existence result

We first prove the following Lemma

Lemma 2 . *Assume* (A_1) *and* (A_5) *hold. Then the index of the compact mapping* Q_T *satisfies the following condition: There exists* $\gamma_1 > 0$ *small enough such that for all*$\gamma \in (-\gamma_1, \gamma_1)$, $\gamma \neq 0$ *such that*

$$i(Q_T, \Omega(O_0, \gamma), j_*(M)) = \chi(C_0).$$

Proof. By the discussion above, and the contraction property

$$i(Q_T, \Omega(O_0, \gamma), j_*(M)) = i(Q_T, B(C_0, \gamma), j_*(M)) = \chi(C_0).$$

Let $\gamma_0 > 0$ be such that (7) doesnot have a T_0-periodic solution with initial condition in $B(C_0, -\gamma_0) \cup B(C_0, \gamma_0)$. We have the following direct consequence of Lemma 2

Corollary 2 . *Assume that* (A_1) *and* $(A_5)'$ *hold. Then,*

$$i(Q_T, \Omega(O_0, \gamma), j_*(M)) = 1$$

for every $-\gamma_0 < \gamma < \gamma_0$ *with* $\gamma \neq 0$.

Proof of Theorem 2. Consider the following change of variable in (3)

$$u(t) = \omega(t, 0, v(t)). \tag{23}$$

For every $v(\cdot) \in M$, (23) defines $u(\cdot)$ uniquely with initial condition $u(0) = \omega(0, 0, v(0)) = v(0)$. Hence, u is a solution of (3) if and only if

$$\omega'_{(1)}(t, 0, v(t)) \oplus \omega'_{(3)}(t, 0, v(t))v'(t) = f(\omega(t, 0, v(t))) \oplus \beta \odot \psi(t, \omega(t, 0, v(t))).$$

By the definition of $\omega(t, 0, v(t))$,

$$\omega'_{(1)}(t, 0, v(t)) = f(\omega(t, 0, v(t)))$$

Thus, by (A_3)

$$\begin{aligned} v'(t) &= (\omega'_{(3)}(t, 0, v(t))^{-1} \odot \beta \odot \psi(t, 0, v(t)) \\ &= \beta \odot \Psi(t, v(t)). \end{aligned} \tag{24}$$

Let t_0 be the pasting point of Remark 2. Integrating (24) for $t \in (0, T)$, we get

$$v(t) = v(0) \oplus \beta \int_0^t \Psi(s, v(s))\, ds \tag{25}$$

if $0 < t < t_0$, or

$$v(t) = v(0) \oplus \beta \odot \int_0^{t_0} \Psi(s, v(s))\, ds - (-\beta) \odot \int_{t_0}^t \Psi(s, v(s))\, ds, \tag{26}$$

if $t_0 < t < T$. Let $u(\cdot)$ be an arbitrary T-periodic solution of (3), then, using the change of variable (23), we get

$$v(0) = u(0) = u(T) = \omega(T, 0, v(T)).$$

Hence, $u(t) = \omega(t, 0, v(t))$ is a T-periodic solution of (3) if and only if

$$\begin{aligned} v(t) &= \omega(T, 0, v(T)) \oplus \beta \odot \int_0^t \Psi(s, v(s)), \quad \text{if} \quad 0 < t < t_0 \\ &= \omega(T, 0, v(T)) \oplus \beta \odot \int_0^{t_0} \Psi(s, v(s))\, ds \\ &\quad -(-\beta) \odot \int_{t_0}^t \Psi(s, v(s))\, ds, \qquad \text{if} \qquad t_0 < t < T. \end{aligned} \tag{27}$$

Applying the embedding function j_*, we have

$$\begin{aligned} j_*(v(t)) &= j_*(\omega(T, 0, v(T)) + \beta \int_0^t j_*(\Psi(s, v(s)))\, ds, \quad \text{if} \quad 0 < t < t_0 \\ &= j_*(\omega(T, 0, v(T)) + \beta \int_0^{t_0} j_*(\Psi(s, v(s))\, ds \\ &\quad -\beta \int_{t_0}^t \tilde{j}_*(\Psi(s, v(s)))\, ds, \quad \text{if} \qquad t_0 < t < T. \end{aligned}$$

Hence, the problem of existence of solution of the integral equation (27) is equivalent to the problem of existence of the fixed point of a compact mapping $\Phi_\beta : \Omega_0 \subset j_*(M) \to j_*(M)$ defined by

$$\begin{aligned} \Phi_\beta(j_*(v(t)) &= j_*(\omega(T,0,v(T))) + \beta \int_0^t j_*(\Psi(s,v(s)))\,ds, \quad \text{if } 0 < t < t_0 \\ &= j_*(\omega(T,0,v(T))) + \beta \int_0^{t_0} j_*(\Psi(s,v(s))\,ds \\ &\quad -\beta \int_{t_0}^t \tilde{j}_*(\Psi(s,v(s)))\,ds, \qquad \text{if} \qquad t_0 < t < T. \end{aligned} \tag{28}$$

Since $j_*(M)$ is an absolute retract, it will suffice to show that the compact mapping Φ_β satisfies the following conditions (see [3] chap.6)

(a) $Fix(\Phi_\beta) \cap \partial\Omega_0 = \emptyset$;

(b) $i(\Phi_\beta, \Omega_0, j_*(M)) \neq 0$.

Define the constants:

$$\begin{aligned} M_0 &= \max_{j_*(v(0)) \in \overline{\Omega_0}} \| j_*(v(0)) - Q_T(j_*(v(0)) \| \\ M_1 &= \min_{\zeta \in \partial\Omega_0} \| \zeta - Q_T(\zeta) \|; \\ M_2 &= \max_{\overline{\Omega_0}} \| \frac{d}{d\tau} Q_T((\tau)) \|; \\ M_3 &= \max\{\| j_*\Psi(t,v(t)) \|,\, (t,v) \in [O,T] \times j_*^{-1}(\overline{\Omega}_0)\} \end{aligned}$$

and

$$\beta_0 = \min\{\frac{M_1}{TM_3(2M_2+1)+M_1}, \frac{M_1}{TM_3(2M_2+1)+M_0}\}.$$

For every $0 < \beta < \beta_0$, assume that there exists $v \in j_*^{-1}(\partial\Omega_0)$ such that

$$j_*(v) = \Phi_\beta(j(v(t))), \;\; t \in [0,T].$$

Since $\| \tilde{j}(u) \| = \| j(u) \|$, we have

$$\begin{aligned} \| j_*(v(t)) - Q_T(j_*(v(t))) \| &\leq \| j_*(v(t)) - j_*(\omega(T,0,v(T)) \| \\ &\quad + \| j_*(\omega(T,0,vT)) - Q_T(j_*(v(t))) \| \\ &\leq 2\beta \int_0^T \| j_*(\Psi(t,v(t))) \| \; dt + \| j_*(v(0)) - Q_T(j_*(v(t))) \| \\ &\leq 2\beta \int_0^T \| j_*(\Psi(t,v(t))) \| \; dt + \| j_*(v(0)) - Q_T(j_*(v(0))) \| \\ &\quad + \| Q_T j_*(v(0)) - Q_T j_*(v(t)) \| \\ &\leq 2\beta_0 T M_3 + \beta_0 T M_3 M_2 + M_0 < M_1, \end{aligned}$$

and this is a contradiction. Hence $Fix(\Phi_\beta) \cap \partial\Omega_0 = \emptyset$, $\forall\, 0 < \beta < \beta_0$. Define the homotopy

$$H : j_*(M) \times [0,1] \to j_*(M)$$

by

$$H(j*(v(t)), \lambda) = (1-\lambda)Q_T(j_*(v(t)) + \lambda\Phi_\beta(j_*(v(t)))$$

and assume that there exist$\beta \in (0, \beta_0)$ and$v_\beta(\cdot) \in j_*^{-1}(\partial\Omega_0)$ such that $j_*(v_\beta(t)) = H(j_*(v_\beta(t)), \lambda)$, then

$$\begin{aligned} \| j_*(v_\beta(t)) - Q_T(j_*(v_\beta(t)) \| &= \lambda \| Q_T(j_*(v_\beta(t)) - \Phi_\beta(j_*(v_\beta(t))) \| \\ &\leq 2\beta \int_0^T \| j_*(\Psi(t, v_\beta(t))) \| + \| j_*(v(0))) - Q_T j_*(v_\beta(t)) \| \end{aligned}$$

Using similar arguments as before, we obtain a contradiction. Thus, $H(\cdot, \lambda)$ is an admissible homotopy, and by the homotopy invariance property , we have

$$i(H(\cdot, 0), \Omega_0, j_*(M)) = i(H(\cdot, 1), \Omega_0, j_*(M)),$$

that is

$$i(Q_T, \Omega_0, j_*(M)) = i(\Phi_\beta, \Omega_0, j_*(M)) \tag{29}$$

and by Lemma 2 and (A_5),

$$i(\Phi_\beta, \Omega_0, j_*(M)) \neq 0.$$

4 Acknowledgement

The author thanks the referee for his valuable remarks and suggestions.

References

[1] B. Bede and S.G. Gal, Generalization of the differentiability of fuzzy number-valued functions with application to fuzzy differential equations, Fuzzy sets and Systems, 151(2005) 518-599.

[2] A. Capieto, J. Mawhin and F. Zanolin, Continuation theorems for periodic pertubation of autononous systems, Trans. Amer. Math. Soc., 329(1992), 41-72.

[3] J. Cronin, Lapounov stability and periodic solutions, Bol. Soc. Mat. Mexicana, (1965)22-27.

[4] J. Cronin, The point at infinity and periodic solutions, J. Diff. Equations 1(1965), 156-170.

[5] J. Cronin, Periodic solutions of some non linear differential equations, J. Diff. Equations "(1967) 31-46.

[6] K. Deimling, Non linear functional analysis, Springer Verlag, Berlin, 1984.

[7] A. Halanay, Solutions périodiques des systemes non linéaires a petit parametre, Ren. Accad. Naz. Lincei, 22(Ser.8), (1957), 30-32.

[8] M. Kamenskii, O. Makarenkov, P. Nistri, Periodic solutions of periodically pertubed planar autonomous systems: A topological approach, Advances in differential equations, 11, 4(2006), 399-418.

[9] M.A. Krasnosel'skii and P.P. Zabreiko, Geometrical methods of non linear analysis, Sringer Verlag, Berlin, 1984.

[10] A. Lando, Periodic solutions of non linear systems with forcing term, J. Diff. Equations 3(1971), 262-279.

[11] A. Lando, Forced oscillations of two-dimensional non linear systems, Applicable Analysis, 28(1988), 285-295.