SJPAM, Vol. 3, pp 58 - 65, 2008

SAMSA Journal of Pure and Applied Mathematics

On the onset of convection in a porous layer in the presence of Dufour and Soret effects

S.S. Motsa
University of Swaziland, Mathematics Department
Private Bag 4, Kwaluseni, Swaziland

Abstract

This paper addresses the problem of double-diffusive convection in a horizontal layer filled with a fluid in the presence of temperature gradients (Soret effects) and concentration gradients (Dufour effects). The onset of convection is studied using linear stability analysis. The critical Rayleigh numbers for the onset of convection are determined in terms of the governing parameters.

1 Introduction

The study of the onset of convection in porous medium is of paramount importance to the study of the behaviour of fluids in the crust of the earth, in geology, geophysics, metallurgy, material science and petroleum engineering. A detailed review of the topic is given by Nield and Bejan (2006).

Convection motion sets in if a quiescent fluid layer bounded by horizontal boundaries is subjected to an adverse temperature gradient (heated from below). The convective stability limit of a fluid heated from below is described by the Rayleigh number which has been determined to be of order $4\pi^2$ (Nield 1991). This limit determines the point at which convective instability may start occurring in a fluid layer. Convective fluid flow induced by buoyancy forces resulting from the imposition of thermal and solutal boundary conditions is called double-diffusive convection.

Taslim and Nasurawa (1986) and Malashetty (1993) used linear stability analysis to investigate the onset of convection in a double-diffusive flow. This work was later extended by Nield *et. al* (1993) to consider the effects of inclined temperature and solutal gradients. In this study it was observed that both the thermal and solutal Rayleigh numbers contributed significantly in the onset of the convective instability. More recently Bahloul *et. al* (2003) investigated the effect of thermal diffusion (Soret effect) in double-diffusive flows. Soret effect refers to the maintenance of concentration gradient due to temperature gradient. Maintenance of temperature gradient due to concentration gradient is called the Dufour effect.

In this study we investigate the effect of both the Soret and Dufour effects on the onset of double-diffusive convection. Whilst Soret effects have been widely studied in related convective studies in porous media (see for example, Ouarzazi and Bois 1994, Bahloul *et. al* 2003), the Dufour effects has received little attention. This is partly due to the fact the Dufour effects have been reported to be negligible in liquid mixtures (Schechter *et. al* 1972). In gas mixtures, however, the Dufour effects are very important.

2 Mathematical Formulation

We consider a porous medium occupying a horizontal layer of fluid mixture of height H as shown in Figure 1. A constant temperature and concentration distribution is prescribed at the boundaries. The vertical temperature difference across the boundaries is ΔT and the vertical concentration difference is ΔC.

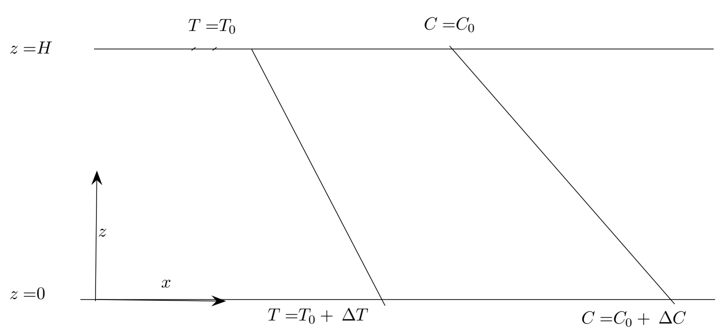

Figure 1: Horizontal layer heated and salted from below

We assume that the medium is homogeneous and isotropic and that Darcy's law is valid and the Oberbeck-Boussinesque approximation is applicable. Accordingly, the appropriate governing equations are

$$\nabla \cdot \mathbf{v} = 0, \tag{1}$$

$$0 = -\nabla \mathbf{p} - \frac{\mu}{K}\mathbf{v} + \rho_f \mathbf{g}, \tag{2}$$

$$\sigma \frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = \alpha_m \nabla^2 T + D_{TC} \nabla^2 C, \tag{3}$$

$$\phi \frac{\partial C}{\partial t} + \mathbf{v} \cdot \nabla C = D_m \nabla^2 C + D_{CT} \nabla^2 T, \tag{4}$$

$$\rho_f = \rho_0 [1 - \gamma_T (T - T_0) - \gamma_C (C - C_0)]. \tag{5}$$

Here $(u, v, w) = \mathbf{v}$ is the Darcy velocity, $\mathbf{p}$ is the pressure, $\mathbf{g}$ is the acceleration due to gravity, μ is the viscosity, T and C are the temperature and concentration respectively. The subscripts m and f refer to the porous medium and the fluid, respectively. Also $\sigma = \dfrac{(\rho c_p)_f}{(\rho c_p)_m}$ where ρ and c_p denote density and specific heat while K and ϕ are the permeability and porosity of the medium, α_m and D_m are thermal conductivity and solutal diffusivity of the medium, D_{TC} and D_{CT} are the Dufour and Soret coefficients. Also γ_T and γ_C are the thermal and solutal expansion coefficients in the medium.

As shown in Figure 1, the boundary conditions on C and T are

$$C = C_0 + \Delta C \,, \quad T = T_0 + \Delta T \quad \text{at} \quad z = 0 \tag{6}$$

$$C = C_0 \,, \quad T = T_0 \quad \text{at} \quad z = H. \tag{7}$$

Also, since the boundaries are impermeable, we have

$$\mathbf{v} \cdot \hat{\mathbf{n}} = 0. \tag{8}$$

For the set of equations (1 - 8), the steady state solution is given as

$$V_s = (0, 0, 0), \tag{9}$$

$$T_s = T_0 + \Delta T \left(1 - \frac{z}{H}\right), \tag{10}$$

$$C_s = C_0 + \Delta C \left(1 - \frac{z}{H}\right), \tag{11}$$

$$p_s = p_0 - \rho_0 g \left[(T_0 + C_0) z + (\gamma_T \Delta T + \gamma_C \Delta C)\left(z - \frac{z^2}{2H}\right)\right]. \tag{12}$$

We now superimpose small perturbations on the basic state in the form

$$\mathbf{v} = \mathbf{v}' \,, \quad T = T_s + T' \,, \quad \mathbf{p} = \mathbf{p}_s + \mathbf{p}' \,, \quad C = C_s + C' \tag{13}$$

where the primes denote the perturbed quantities. Substituting (13) into (1) - (4) and neglecting higher order terms of the perturbed quantities we obtain,

$$\nabla \cdot \mathbf{v}' = 0, \tag{14}$$
$$0 = -\nabla \mathbf{p}' - \frac{\mu}{K}\mathbf{v}' - \rho_0(\gamma_T T' + \gamma_C C')\mathbf{g}, \tag{15}$$
$$\sigma \frac{\partial T'}{\partial t} - w' \frac{\Delta T}{H} = \alpha_m \nabla^2 T' + D_{TC} \nabla^2 C', \tag{16}$$
$$\phi \frac{\partial C'}{\partial t} - w' \frac{\Delta C}{H} = D_m \nabla^2 C' + D_{CT} \nabla^2 T', \tag{17}$$

We non-dimensionalize equations (14 - 17) by introducing the following dimensionless variables,

$$\begin{array}{ccc} (x^*, y^*, z^*) = \frac{1}{H}(x, y, z) & \mathbf{v}^* = \frac{H}{\alpha_m}\mathbf{v}' & t^* = \frac{\alpha_m}{\sigma H^2} t \\ T^* = \frac{T'}{\Delta T} & C^* = \frac{C'}{\Delta C} & \mathbf{p}^* = \frac{K}{\mu \alpha_m} p'. \end{array} \tag{18}$$

The dimensionless equations are, with "*" omitted for brevity,

$$\nabla \cdot \mathbf{v} = 0, \tag{19}$$
$$\nabla \mathbf{p} = \mathbf{v} + Ra(T + NC) = 0, \tag{20}$$
$$\frac{\partial T}{\partial t} - w = \nabla^2 T + D_f \nabla^2 C, \tag{21}$$
$$\frac{\phi}{\sigma}\frac{\partial C}{\partial t} - w = \frac{1}{Le}\nabla^2 C + S_r \nabla^2 T. \tag{22}$$

where

$$Ra = \frac{g\gamma_T K \Delta T}{\nu \alpha_m} \quad \text{is the thermal Rayleigh number}$$
$$N = \frac{\gamma_C \Delta C}{\gamma_T \Delta T} \quad \text{is the buoyancy ratio}$$
$$Le = \frac{\alpha_m}{D_m} \quad \text{is the Lewis number}$$
$$D_f = \frac{D_{TC}\Delta C}{\alpha_m \Delta T} \quad \text{is the Dufour parameter}$$
$$S_r = \frac{D_{CT}\Delta T}{\alpha_m \Delta C} \quad \text{is the Soret parameter}$$

The pressure perturbations are removed by taking the curl of (20). Taking only the z component, the resulting equation becomes

$$\nabla^2 w = Ra \nabla_1^2 (T + NC) \tag{23}$$

where

$$\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

The boundary conditions become

$$w = T = C = 0 \quad \text{at} \quad z = 0,\ 1 \tag{24}$$

3 Linear Stability Analysis

We consider sinusoidal expansions of the form

$$(W, T, C) = (\tilde{W}, \tilde{T}, \tilde{C})e^{ilx+imy+st} \tag{25}$$

where l and m are dimensionless wave numbers and s is the growth rate. Substituting (25) into (21 - 23) results in the following equations

$$(D^2 - \alpha^2 - s)\tilde{T} + D_f(D^2 - \alpha^2)\tilde{C} + \tilde{W} = 0, \tag{26}$$

$$(D^2 - \alpha^2)\tilde{W} + \alpha^2 Ra(\tilde{T} + N\tilde{C}) = 0, \tag{27}$$

$$\left[\frac{1}{Le}(D^2 - \alpha^2) - \frac{\phi}{\sigma}s\right]\tilde{C} + S_r(D^2 - \alpha^2)\tilde{T} + \tilde{W} = 0 \tag{28}$$

where

$$D^2 = \frac{d^2}{dz^2} \quad \text{and} \quad \alpha^2 = l^2 + m^2.$$

The boundary conditions are

$$\tilde{W} = \tilde{T} = \tilde{C} = 0 \quad \text{at} \quad z = 0\ ,\quad 1. \tag{29}$$

Solutions of the form

$$(\tilde{W}, \tilde{T}, \tilde{C}) = (\tilde{W}_0, \tilde{T}_0, \tilde{C}_0)\sin j\pi z \tag{30}$$

are possible if

$$\begin{aligned} &J(J+s)(J+\Phi s) - LeD_fS_rJ^3 = \\ &\alpha^2 Ra(J + \Phi s - D_f LeJ) + Ra_s\alpha^2(J + s - S_rJ) \end{aligned} \tag{31}$$

where $J = j^2\pi^2 + \alpha^2$, $\Phi = \frac{\phi}{\sigma}Le$, $Ra_s = NLeRa = \dfrac{g\gamma_T K\Delta C}{\nu D_m}$ is the solutal Rayleigh number.

4 Results and Discussion

At marginal stability $s = i\omega$ where ω is real, and the real and imaginary parts of equation (31) yields

$$J^2(1 - LeD_fS_r) - \Phi\omega^2 = \alpha^2[Ra(1 - D_fLe) + Ra_s(1 - S_r)], \quad (32)$$
$$\omega[J^2(1 + \Phi) - (\Phi Ra + Ra_s)\alpha^2] = 0. \quad (33)$$

From equation (32) if $\omega = 0$, then

$$Ra + Ra_s\frac{(1 - S_r)}{1 - D_fLe} = \frac{J^2(1 - LeD_fS_r)}{\alpha^2(1 - D_fLe)} \quad (34)$$

which represents the boundary for monotonic or stationary instability. In particular, to find the lowest threshold of instability as a function of α we set $\frac{\partial R_a}{\partial \alpha} = 0$. This gives $\alpha_c = \pi$ and we conclude that the critical instability Rayleigh number is

$$Ra_c = Ra_s\frac{S_r - 1}{1 - D_fLe} + \frac{4\pi^2(1 - LeD_fS_r)}{1 - D_fLe} \quad (35)$$

If $S_r = D_f = 0$ we get

$$Ra_c + Ra_s = 4\pi^2$$

which is the boundary for stationary instability reported in Nield (1999) for double-diffusive convection in the absence of Soret and Dufour effects. In the absence of solute we have $Ra_s = 0$ and the critical Rayleigh number in this case is given by $Ra_c = 4\pi^2$ which is the exact result previously reported previously by Nield (1991).

From the result in equation (35) it can be seen that in the presence of a solute (i.e when $Ra_s \neq 0$), if $D_f = 0$ and $S_r > 1$ the critical Rayleigh number will increase when the Soret number increases. This means that the Soret effect serves to delay the onset of convection, i.e is stabilizes the flow. On the other hand if $S_r = 0$ and $D_f > \frac{1}{Le}$ we see from equation (35) that the critical Rayleigh number will decrease when the Dufour number is increased. Thus the Dufour effect has a destabilizing effect.

If $\omega \neq 0$, we obtain the critical Rayleigh number for the onset of overstability (oscillatory convection) from equation (30)

$$\Phi Ra^{over} + Ra_s = 4\pi^2(1 + \Phi) \quad (36)$$

which is exactly equal to that of Nield (1999). We note that this result does not depend on the Soret and Dufour parameters. This means that overstability

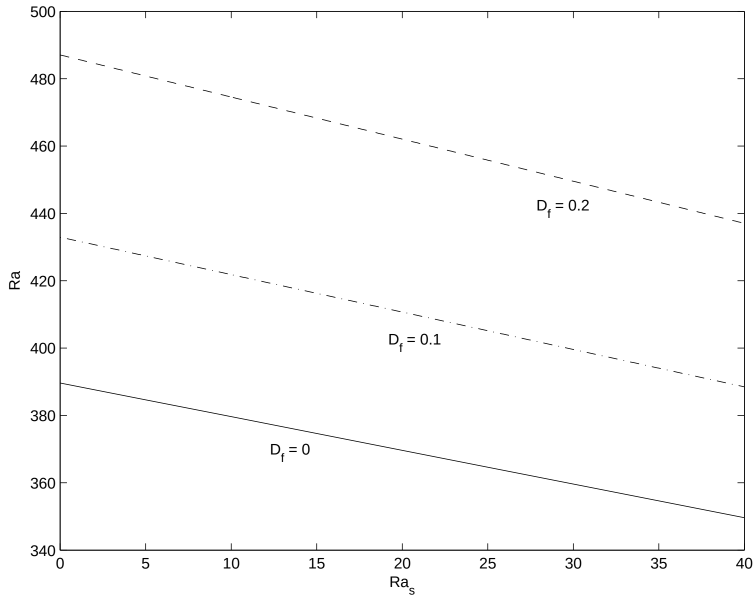

Figure 2: Effect of varying D_f on the onset of stability

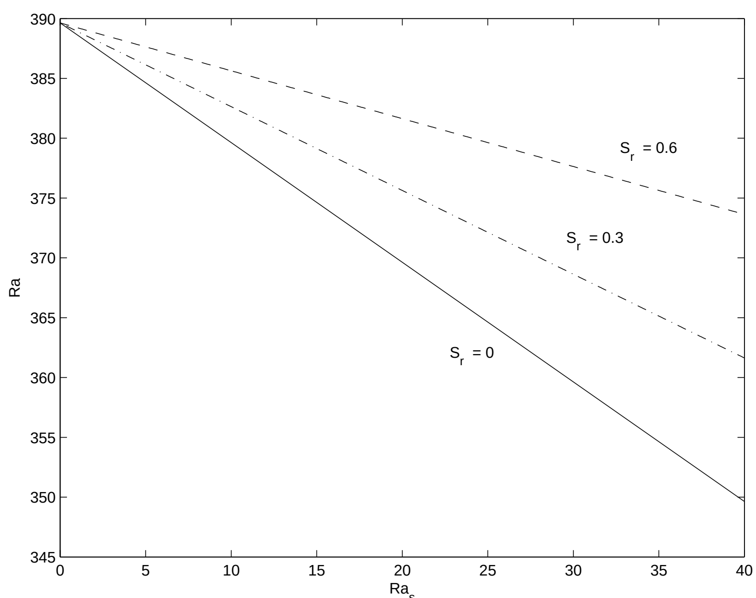

Figure 3: Effect of varying S_r on the onset of stability

is not affected by the presence of Soret and Dufour effects in double-diffusive flow.

Figure 2 and Figure 3 show the effects of varying the Dufour and Soret parameters, respectively, on the onset of stationary convection. The diagrams indicate that the Dufour and Soret effects delay the onset of convection.

5 Conclusion

In this study we used linear stability analysis to investigate cross-diffusion (Soret and Dufour) effects in double-diffusive (thermal and solutal gradients imposed) convection in a fluid-saturated porous medium. The aim of this work was to investigate the Soret and Dufour effects on the onset of convection. It was found that, in the case of stationary instability, the Soret effect had a stabilizing effect whereas the Dufour effect was destabilizing. The cross-diffusion effects were found to have no effect on overstability. In the limiting case when the Soret and Dufour parameters were set to be equal to zero the results presented in this study reduced to those reported in previous studies on related double-diffusive convective flow.

References

[1] A. Bahloul, N. Boutana, P. Vasseur, Double-diffusive and Soret-induced convection is a shallow horizontal porous layer. *J. Fluid Mech.* **491** (2003) 325.

[2] M.S. Malashetty, Anisotropic thermoconvective effects on the onset of double diffusive convection in a porous medium, *Intl. J. Heat and Mass Transfer*, **39** (1993) 2397.

[3] D.A. Nield, A. Bejan, Convection in Porous Media, third ed., Springer-Verlag, New York, 2006.

[4] D.A. Nield, Convection in a porous medium with inclined temperature gradient, Int. J. Hear Mass Transfer, **34**, (1991), 87.

[5] D.A. Nield, D.M. Manole, J.L. Lage, Convection induced by inclined thermal and solutal gradients in a shallow horizontal layer of a porous medium, *J.Fluid. Mech*, **257** (1993), 559.

[6] D.A. Nield, A. Bejan, Convection in Porous Media, second ed., Springer-Verlag, New York, 1999.

[7] Ouarzazi, M.N., Bois, P.A., Convective instability of a fluid mixture in a porous medium with time-dependent temperature. *Eur. J. Mech. B/Fluids.* **13** (1994) ,275.

[8] D. Poulikakos, Double diffusive convection in a horizontal sparcely packed porous layer. *Intl. Commun. Heat Mass Transfer*, **13** (1986) 587.

[9] Schechter, R.S., Prigogine,I. and J.R. Hamm, Thermal diffusion and convective stability, *Phys. Fluids*,**15**, (1972) 379.

[10] M.E. Taslim, U. Narusawa, Binary fluid convection and double-diffusive convection in porous medium, *J. Heat Transfer* **108** (1986) 221.