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COMBINED SINGULAR AND IMPULSE CONTROL FOR JUMP DIFFUSIONS

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[1] E. Chikodza

Abstract

In this work we present and investigate the combined singular and impulse control problem for jump diffusions. Such problems frequently arise in finance, for instance, when both fixed and proportional transaction costs are considered. A verification theorem for the generalised combined singular and impulse control is formulated and proved. The verification theorem provides sufficient conditions for the existence of both the value function and optimal combined controls. An example is presented to illustrate the application of the theory.

Keywords: *combined singular and impulse control, jump diffusion, variational inequalities*

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[1]Department of Natural Sciences,Great Zimbabwe University,
P.O. Box 1235, Masvingo, **Zimbabwe**.
e-mail: egchikodza@yahoo.co.uk

1 Introduction

The problem of portfolio optimization in the presence of transaction costs has been explored by several researchers before, see, for example, [1], [5] and [6]. The inclusion of both fixed and proportional transaction costs in a portfolio optimization question gives rise to a problem which exhibits both singular and impulse control features. To the best of our knowledge the theory of combined singular and impulse control for diffusion processes was presented for the first time in [8].

In this paper we extend the theory treated in [8] to the jump diffusion context and again, as far as we know, this has not been done before. That is to say, using some of the arguments presented in [8] we develop the theory of combined singular and impulse control for Lévy processes. An additional feature which distinguishes this paper from any other previous work is that it illustrates the application of combined singular impulse control to the problem of optimal harvesting, with density dependent prices, in a jump diffusion set up and in the presence of transaction costs.

Our example on optimal harvesting with density dependent prices is motivated by Example 3.1 in [2]. In [2] the problem of optimal stochastic harvesting with density- dependent prices for diffusions is discussed under the *no transaction costs* assumption. For an extensive coverage of the theory and application of singular control and impulse control as separate stochastic control techniques for Lévy processes see, for example, [11] and references given therein.

The rest of this paper is organised as follows. In Section 2 the general combined singular and impulse control problem is formulated. In Section 3 the verification theorem and its proof are presented. An example on the application of the theory of combined singular and impulse control for jump diffusions is discussed in Section 4. In this example we take both proportional and transaction costs into account. The paper has some conclusive remarks that are presented in Section 5.

We now present the formulation of the combined singular and impulse control problem for Lévy processes.

2 Problem Formulation.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a filtered complete probability space satisfying the usual conditions.

It is assumed that in the absence of interventions the state, $Y(t) \in \Re^k$, of a given system evolves according to the following equations

$$dY(t) = b(Y(t))dt + \sigma(Y(t))dB(t) + \int_{\Re^d} \gamma(Y(t^-), z)\tilde{N}(dt, dz); \qquad (1)$$

$$Y(0^-) = y \in \Re^k, \qquad (2)$$

where $b : \Re^k \to \Re^k,\ \sigma : \Re^k \to \Re^{k\times m}$ and $\gamma : \Re^k \times \Re^d \to \Re^{k\times d}$ are given functions satisfying the conditions for the existence and uniqueness of a strong solution, $Y(t)$. For details concerning such conditions see, for example, Theorem 1.19 in [11]. Here, $B(t)$ is m- dimensional Brownian motion with respect to $\{\mathcal{F}_t\}$ and $\tilde{N}_r(.,.)$ is a compensated Poisson random measure given by

$$\tilde{N}_r(dt, dz) = N_r(dt, dz) - dt\nu_r(dz); \qquad r = 1, 2, . . ., \ d$$

where $\nu_r(.)$ is a Levy measure associated with the Poisson random measure $N_r(.,.)$. For a more extensive treatment of random measures and stochastic differential equations with a jump component see, for example [4], [7], [9], [11] and [12].

The generator of $Y(t)$ coincides with the second order integro-differential operator $\mathcal{L}$, given by

$$\begin{aligned}\mathcal{L}\phi(y) &= \sum_{i=1}^{k} b_i\Big(y, u(y)\Big)\frac{\partial \phi}{\partial y_i} + \frac{1}{2}\sum_{i,s=1}^{k}(\sigma\sigma^T)_{is}\Big(y, u(y)\Big)\frac{\partial^2 \phi}{\partial y_i \partial y_s}\\ &+ \int_{\Re^d}\sum_{r=1}^{d}\{\Big(\phi(y + \gamma^{(r)}(y, u(y), z) - \phi(y) - \nabla\phi(y)^T\gamma^{(r)}\Big(y, u(y), z\Big)\}\nu_r(dz_r).\end{aligned} \qquad (3)$$

Suppose that at any given point τ_j the decision maker is free to give the system an *impulse*, $\xi_j \in \mathcal{Z} \subset \Re^p$, where $\mathcal{Z}$ is the set of all admissible impulses. At this juncture it is necessary to define the notion of an *impulse control.*

Definition 2.1 *(Impulse Control) An impulse control, for the system described by* (2.1) – (2.2)*, is a double sequence*

$$v = (\tau_1, \tau_2, \ . \ . \ ., \tau_j . \ . \ ., \xi_1, \xi_2, \ . \ . \ ., \xi_j, \ . \ . \ .)_{j \leq M} \quad M \leq \infty$$

where $0 \leq \tau_1 \leq \tau_2 \leq . \ . \ .$ *is an increasing sequence of* $\mathcal{F}_t$*-stopping times (intervention times) and* $\xi_1, \xi_2, . \ . \ .$ *are the corresponding* $\mathcal{F}_{\tau_j}$*-adapted impulses (interventions) at these stopping times.*

We now consider that as a result of applying an impulse control $v = (\tau_1, \tau_2, \ . \ . \ ., \tau_j \ . \ . \ ., \xi_1, \xi_2, \ . \ . \ ., \xi_j \ . \ . \ .)$, the corresponding controlled state process, $Y^{(v)}(t)$, evolves according to $(2.4) - (2.7)$ stated below

$$Y^{(v)}(0^-) = y \text{ and } Y^{(v)}(t) = Y(t); \qquad 0 < t < \tau_1 \tag{4}$$

$$Y^{(v)}(\tau_j) = \Gamma(\check{Y}^{(v)}(\tau_j^-), \xi_j); \qquad j = 1, 2, \ . \ . \ . \tag{5}$$

$$dY^{(v)}(t) = b(Y^{(v)}(t))dt + \sigma(Y^{(v)}(t))dB(t) + \int_{\Re^d} \gamma(Y^{(v)}(t^-), z)\tilde{N}(dt, dz) \tag{6}$$

$$\text{for } \tau_j < t < \tau_{j+1} \tag{7}$$

where

$$\check{Y}^{(v)}(\tau_j^-) = Y^{(v)}(\tau_j^-) + \triangle_N Y(\tau_j), \tag{8}$$

represents the jump in $Y^{(v)}(\tau_j)$ which sterms from $N(.,.)$ and

$$\Gamma : \Re^k \times \mathcal{Z} \to \Re^k$$

is a given function. See for examle [11] for more details on the general formulation of the impulse control problem.

Let $\mathcal{S} \subset \Re^k$ be a fixed Borel set in which we seek solutions to the problem such that $\mathcal{S} \subset \bar{\mathcal{S}}^0$. Here $\mathcal{S}^0$ denotes the interior of $\mathcal{S}$ and $\bar{\mathcal{S}}^0$ is the closure of $\mathcal{S}^0$.

Suppose we are given continuous functions $f : \mathcal{S} \to \Re, \ g : \Re^k \to \Re$, $\kappa = [\kappa_{ie}] \in \Re^{k \times p}$ and $\theta = [\theta_i]$. Additionally, let the profit of making an intervention with impulse $\xi \in \mathcal{Z}$ when the state is y be $K(y, \xi)$, where $K : \mathcal{S} \times \mathcal{Z} \to \Re$. Let $\mathcal{V}$ be the set of *admissible impulse controls*, v, and also assume that

$$E^y\Big[\int_0^{\tau_{\mathcal{S}}} f^-(Y^{(v)}(t)dt\Big] < \infty \quad \text{for all } y \in \Re^k, \ v \in \mathcal{V}, \tag{9}$$

$$E\Big[g^-(Y^{(v)}(\tau_{\mathcal{S}}))\chi_{\{\tau_{\mathcal{S}} < \infty\}}\Big] < \infty \quad \text{for all } y \in \Re^k, \ v \in \mathcal{V}, \tag{10}$$

$$E\Big[\sum_{\tau_j \leq \tau_{\mathcal{S}}} K(Y^{(v)}(\tau_j^-), \xi_j)\Big] < \infty \quad \text{for all } y \in \Re^k, \ v \in \mathcal{V}. \tag{11}$$

where

$$\tau_{\mathcal{S}} = \inf\{t \geq 0, ; Y^{(v)}(t) \notin \mathcal{S}\}.$$

The notion of *intervention operator* plays a crucial role in the rest of this work, so we define it below.

Definition 2.2 *Let $\mathcal{H}$ be the space of all measurable functions $h : \mathcal{S} \to \Re$. The intervention operator $\mathcal{M} : \mathcal{H} \to \mathcal{H}$ is defined by*

$$\mathcal{M}h(y) = \sup\{h(\Gamma(y, \xi)) + K(y, \xi);\ \xi \in \mathcal{Z}\}. \tag{12}$$

We let

$$\mathcal{T} = \{\tau;\ \tau \text{ stopping times},\ 0 \leq \tau \leq \tau_{\mathcal{S}}\}.$$

Suppose that at times $t_n \in [\tau_j; \tau_{j+1}]$ one is allowed to intervene and apply, whenever it is profitable to do so, the singular control ψ for $n = 1, 2,\ . . .,\ q$. For a more elaborate treatment of the singular control theory and its applications see [10] and [8].

We note that $\psi \in \Re^p$ is an adapted cadlag process with non-negative, increasing components such that $\psi(0^-) = 0$. We also consider $u(t)$ to be an adapted cadlag process (our absolutely continuous control) with values in a given open, connected set $\mathcal{U} \subset \Re^k$. Adopting the notation in [11] we let the jumps caused by the singular control ψ be denoted by

$$\triangle_\psi Y(t) = \kappa(Y(t^-))\triangle_\psi.$$

and interpret

$$\triangle_\psi \phi(Y(t_n)) = \phi(Y(t_n)) - \phi(Y(t_n^-)) \tag{13}$$

as the increase in ϕ due to the jump $\triangle\psi(t) = \psi(t) - \psi(t^-)$ at $t = t_n$. Denote by $\mathcal{W}$ the set of all admissible combined controls $w = (v, u, \psi)$. Suppose that as a consequence of applying the combined singular and impulse control, $w = (v, u, \psi)$, the state process $Y^{(w)}$ satifies (2.14) – (2.17) given as follows

$$Y^{(v)}(0^-) = y \text{ and } Y^{(v)}(t) = Y(t); \qquad 0 < t < \tau_1 \tag{14}$$

$$Y^{(v)}(\tau_j) = \Gamma(\check{Y}^{(v)}(\tau_j^-), \xi_j); \qquad j = 1, 2,\ . . \tag{15}$$

$$dY^{(v)}(t) = b(Y^{(v)}(t), u(t))dt + \sigma(Y^{(v)}(t), u(t))dB(t) +$$

$$+ \int_{\Re^l} \gamma(Y^{(v)}(t^-), u(t), z)\tilde{N}(dt, dz) + \kappa(Y^v(t), u(t))d\psi \tag{16}$$

$$\text{for} \qquad \tau_j < t < \tau_{j+1} < \tau^* \tag{17}$$

where

$$\tau^* = \tau^*(\omega) = \lim_{R\to\infty}(\inf\{t > 0; \mid Y^{(v)}(t) \mid\geq R\}) \leq \infty. \tag{18}$$

Define a performance functional, $J^{(w)}$, for the controlled process $Y^{(w)}$, by

$$\begin{aligned} J^{(w)}(y) &= E^y\Big[\int_0^{\tau_{\mathcal{S}}} f(Y^{(w)}(t), u(t))dt + g(Y^{(w)}(\tau_{\mathcal{S}}))\chi_{\{\tau_{\mathcal{S}}<\infty\}} + \\ &+ \int_0^{\tau_{\mathcal{S}}} \theta^T(Y(t))d\psi(t) + \sum_{\tau_j\leq\tau_{\mathcal{S}}} K(Y^{(w)}(\tau_j^-), \xi_j)\Big]. \end{aligned}$$

The *combined singular and impulse control problem* for jump diffusions is to find $\Phi(y)$ and $w^* \in \mathcal{W}$ such that

$$\Phi(y) = \sup\{J^{(w)}(y); w \in \mathcal{W}\} = J^{(w^*)}(y). \tag{19}$$

In the next section we state and prove a verification theorem for the combined singular and impulse control problem of jump diffusions. The theorem that we present below constitutes the main result of this work.

3 Main Result

Theorem 3.1 *(Hamilton-Jacobi-Bellman Quasi-Variational Inequalities for combined singular and impulse control)*

1. *Suppose that we can find $\phi : \bar{\mathcal{S}} \to \Re$ such that*

   (i) $\phi \in C^2(\mathcal{S}^o) \cap C(\bar{\mathcal{S}})$,

   (ii) $\phi \geq \mathcal{M}\phi$ *on* $\mathcal{S}^0$,

   (iii) $\sum_{i=1}^k \kappa_{ie}(y)\frac{\partial\phi}{\partial y_i}(y) + \theta_e(y) \leq 0$ *for all* $y \in \mathcal{S}$,

   Define

$$D = \{y \in \mathcal{S};\ \max_e\{\mathcal{M}\phi(y) - \phi(y), \sum_{i=1}^k \kappa_{ie}(y)\frac{\partial\phi}{\partial y_i}(y) + \theta_e(y)\} \leq 0\} \tag{20}$$

   Assume that $Y^{(w)}(t)$ spends 0 time on ∂D a.s., that is,

(iv) $E^y\Big[\int_0^{\tau_S} \chi_{\partial D}(Y^{(w)}(t))dt\Big] = 0$ *for all* $y \in \mathcal{S},\ w \in \mathcal{W}$,
and suppose that

(v) ∂D *is a Lipschitz surface,*

(vi) $\phi \in C^2(\mathcal{S} \setminus \partial D)$ *with locally bounded derivatives near* ∂D,

(vii) $\mathcal{L}^w\phi + f(y,w) \leq 0$ *for all* $w \in \mathcal{W}$, $y \in \mathcal{S}^0 \setminus \partial D$,

(viii) $Y^{(w)}(\tau_{\mathcal{S}}) \in \partial S$ *a.s. on* $\{\tau_{\mathcal{S}} < \infty\}$ *and*
$\phi(Y^{(w)}(t)) \to g(Y^{(w)}(\tau_{\mathcal{S}})).\chi_{\{\tau_{\mathcal{S}}<\infty\}}$ *as* $t \to \tau_{\mathcal{S}}^-$ *a.s., for all* $y \in \mathcal{S},\ w \in \mathcal{W}$,

(ix) $\{\phi^-(Y^{(w)}(\tau));\ \tau \in \mathcal{T}\}$ *is uniformly integrable, for all* $y \in \mathcal{S}$, $w \in \mathcal{W}$,

(x) $E^y\Big[\int_0^{\tau_S}\{\mid \sigma^T(Y(t),u(t))\bigtriangledown\phi(Y(t)) \mid^2 + \sum_{m=1}^d \int_{\Re^k} \mid \phi(Y(t)+\gamma^{(m)}) - \phi(Y(t)) \mid^2 \nu_m(dz)\}dt\Big] < \infty$
for all $y \in \mathcal{S},\ w \in \mathcal{W}$.
Then

$$\phi(y) \geq \Phi(y) \quad y \in \mathcal{S}.$$

2. Suppose that, in addition to conditions $1(i) - 1(x)$,

(i) *there exists a function* $\hat{w} = (\hat{v}, \hat{u}, \hat{\psi}) \in \mathcal{W}$ *such that*

$$\mathcal{L}^{\hat{w}}\phi(y) + f(y, \hat{w}(y)) = 0 \text{ for all } y \in D$$

(ii) $Y^{\hat{u},\hat{\psi}}(t) \in \bar{D}$

(iii) $d\hat{\psi}(t) = 0$

(iv) $\sum_{e=1}^p\Big\{\sum_{i=1}^k \kappa_{ie}(y)\frac{\partial\phi}{\partial y_i}(Y(t^-)) + \theta_e\Big\}d\hat{\psi}_e^{(c)} = 0$ *for all* $1 \leq p$
where $\psi_e^{(c)}(t)$ *is the continuous part of* $\psi_e^{(c)}$.

(v) $\triangle_{\check{\psi}}\phi(Y(t_n)) + \sum_{e=1}^p \theta_e(Y(t_n^-))\triangle\check{\psi}_e(t_n) = 0$ *for all jumping times* t_n *of* $\check{\psi}$
and
$\lim_{R\to\infty} E^y\Big[\phi(Y^{\hat{u},\hat{\psi}}(T_R))\Big] = E^y\Big[g(Y^{\hat{u},\hat{\psi}}(T)).\chi_{\{T<\infty\}}\Big]$
where

$$T_R = \min(\tau_{\mathcal{S}}, R) \text{ for } R < \infty.$$

and

(vi) $\hat{\xi}(y) \in Argmax\{\phi(\Gamma(y,.)) + K(y,.)\} \in \mathcal{Z}$ *exists for all* $y \in \mathcal{S}$ *and* $\hat{\xi}(.)$ *is a Borel measurable selection.*
Put $\hat{\tau}_0 = 0$ *and define an impulse control*
$\hat{v} = (\hat{\tau}_1, \hat{\tau}_2, \ldots, \hat{\tau}_j, \ldots; \hat{\xi}_1, \hat{\xi}_2, \ldots, \hat{\xi}_j, \ldots)$
inductively by $\hat{\tau}_{j+1} = \inf\{t > \hat{\tau}_j;\ Y^{(\hat{w}_j)}(t) \notin D\} \wedge \tau_{\mathcal{S}}$ *and* $\hat{\xi}_{j+1} = \hat{\xi}(Y^{(\hat{w}_j)}(\hat{\tau}_{j+1}^-))$ *if* $\hat{\tau}_{j+1} < \tau_{\mathcal{S}}$ *where* $Y^{(\hat{w}_j)}$ *is the result of applying the combined control* $\hat{w}_j := (\hat{v}, \hat{u}, \hat{\psi})$ *to* Y.

(vii) *Let* $\hat{w} := (\hat{v}, \hat{u}, \hat{\psi}) \in \mathcal{W}$ *and also suppose that* $\{\phi^-(Y^{(\hat{w})}(\tau));\ \tau \in \mathcal{T}\}$ *is* Q^y*-uniformly integrable for all* $y \in \mathcal{S}$.
Then

$$\phi(y) = \Phi(y) \qquad \text{for all } y \in \mathcal{S}$$

and

$$\hat{w} \in \mathcal{W} \text{ is an optimal combined singular impulse control.}$$

In the proof of this verification theorem we apply arguments used to prove Theorem 2.1 in [2], Theorem 2.1 in [3] and Theorem 4.1 in [8].

Proof 3.1 *On the basis of an approximation argument (see for example Theorem* 2.11 *in* [8]*, Theorem 10.4.1 in* [10] *) and by also using* $1(iv)-1(vi)$ *of the above stated theorem we can assume that* $\phi \in C^2(\mathcal{S}) \cap C(\bar{\mathcal{S}})$*. Consider an arbitrarily chosen impulse control* $v = (\tau_1, \tau_2, \ldots, \tau_j, \ldots; \xi_1, \xi_2, \ldots, \xi_j, \ldots) \in \mathcal{V}$ *and let* $\tau_0 = 0$*. Applying Itô's generalized formula for semimartingales, see for example* [5] *and* [6] *(page* 74 *Theorem* 33*), between the stopping times* τ_j *and* τ_{j+1} *with* $y \in \mathcal{S}$*, we obtain*

$$\phi(\check{Y}(\tau_{j+1}^-)) - \phi(Y(\tau_j)) = \int_{\tau_j}^{\tau_{j+1}} \mathcal{L}\phi(Y(t)))dt +$$
$$+ \int_{\tau_j}^{\tau_{j+1}} \sum_{i=1}^{k} \frac{\partial \phi}{\partial y_i}(Y(t^-)) \sum_{e=1}^{p} \kappa_{ie}(Y(t^-)) d\psi_e^c(t) + \sum_{\tau_j < t_n < \tau_{j+1}} \triangle_\psi \phi(Y(t_n)) \qquad (21)$$

where $\psi_e^c(t)$ *denotes the continuous part of* $\psi_e(t)$ *and* $\check{Y}(\tau_{j+1}^-)$ *is defined as in* (8)*.*
Taking expectations we get

$$E^y\Big[\phi(\check{Y}(\tau_{j+1}^-))\Big] - E^y\Big[\phi(Y(\tau_j))\Big] = E^y\Big[\int_{\tau_j}^{\tau_{j+1}} \mathcal{L}\phi(Y(t)))dt +$$
$$+ \int_{\tau_j}^{\tau_{j+1}} \sum_{i=1}^{k} \frac{\partial \phi}{\partial y_i}(Y(t^-)) \sum_{e=1}^{p} \kappa_{ie}(Y(t^-)) d\psi_e^c(t) + \sum_{\tau_j < t_n < \tau_{j+1}} \triangle_\psi \phi(Y(t_n))\Big]. \qquad (22)$$

This last equation is equivalent to

$$E^y\Big[\phi(Y(\tau_j))\Big] - E^y\Big[\phi(\check{Y}(\tau_{j+1}^-))\Big] = -E^y\Big[\int_{\tau_j}^{\tau_{j+1}} \mathcal{L}\phi(Y(t)))dt +$$

$$+\int_{\tau_j}^{\tau_{j+1}} \sum_{i=1}^{k} \frac{\partial \phi}{\partial y_i}(Y(t^-)) \sum_{e=1}^{p} \kappa_{ie}(Y(t^-))d\psi_e^c(t) + \sum_{\tau_j<t_n<\tau_{j+1}} \triangle_\psi \phi(Y(t_n))\Big]. \quad (23)$$

Summing up from $j = 0$ to $j = m$ yields

$$\phi(y) + \sum_{j=1}^{m} E^y\Big[\phi(Y(\tau_j)) - \phi(\check{Y}(\tau_j^-))\Big] - E^y\Big[\phi(Y(\tau_{m+1}^-))\Big]$$

$$= -E^y\Big[\int_{0}^{\tau_{m+1}} \mathcal{L}\phi(Y(t)))dt \quad (24)$$

$$+\int_{0}^{\tau_{m+1}} \sum_{i=1}^{k} \frac{\partial \phi}{\partial y_i}(Y(t^-)) \sum_{e=1}^{p} \kappa_{ie}(Y(t^-))d\psi_e^c(t) + \sum_{0<t_n<\tau_{j+1}} \triangle_\psi \phi(Y(t_n))\Big]. \quad (25)$$

It is easy to note that

$$\phi(Y(\tau_j)) \leq \phi(\Gamma(Y(\tau_j^-), \xi_j)) + K(Y(\tau_j^-), \xi_j). \quad (26)$$

Applying the definition of the intervention operator, $\mathcal{M}$, we obtain

$$\phi(Y(\tau_j)) = \phi(\Gamma(Y(\tau_j^-), \xi_j)) + K(Y(\tau_j^-), \xi_j) \leq \mathcal{M}\phi(Y(\tau_j^-)) \quad if\ \tau_j < \tau_{\mathcal{S}} \quad (27)$$

and

$$\phi(Y(\tau_j)) = \phi(Y(\tau_{\mathcal{S}}) \quad if\ \tau_j = \tau_{\mathcal{S}}. \quad (28)$$

Thus,

$$\phi(Y(\tau_j)) \leq \phi(\Gamma(Y(\tau_j^-), \xi_j)) \leq \mathcal{M}\phi(Y(\tau_j^-)) - K(Y(\tau_j^-), \xi_j) \quad if\ \tau_j < \tau_{\mathcal{S}} \quad (29)$$

and

$$\phi(Y(\tau_j)) = \phi(Y(\tau_{\mathcal{S}}) \quad if\ \tau_j = \tau_{\mathcal{S}}. \quad (30)$$

From (27) we get

$$\mathcal{M}\phi(Y(\tau_j^-)) - \phi(Y(\tau_j^-)) \geq \phi(Y(\tau_j)) - \phi(Y(\tau_j^-)) + K(Y(\tau_j^-), \xi_j). \quad (31)$$

Applying the mean value theorem we obtain

$$\triangle_\psi \phi(Y(t_n)) = \nabla\phi(Y(t_n))^T \triangle_\psi(Y(t_n))$$

$$= \sum_{i=1}^{k} \sum_{l=1}^{p} \frac{\partial \phi}{\partial y_i} Y(t_n^-)\kappa_{il}(Y(t_n^-))(\triangle_{\xi_l} t_n) \quad (32)$$

Now, combining (31) *and* (32) *results in*

$$\phi(y)+\sum_{j=1}^{m}E^{y}\Big[\{\mathcal{M}\phi(Y(\tau_j^-))-\phi(Y(\tau_j^-))\}\chi_{\{\tau_j<\tau_{\mathcal{S}}\}}\Big]$$
$$\geq E^{y}\Big[\phi(Y(\tau_{m+1}^-))-\int_0^{\tau_{m+1}}\mathcal{L}\phi(Y(t)))dt$$
$$-\int_0^{\tau_{m+1}}\sum_{i=1}^{k}\frac{\partial\phi}{\partial y_i}(Y(t^-)\sum_{e=1}^{p}\kappa_{ie}(Y(t^-))d\psi_e^{(c)}(t)-\sum_{0<t_n<\tau_{j+1}}\triangle_{\psi}\phi(Y(t_n))+$$
$$+\sum_{i=1}^{k}K(Y(\tau_j^-),\xi_j)\Big]\geq E^{y}\Big[\int_0^{\tau_{m+1}}f(Y(t),u(t)))dt+\phi(Y(\tau_{m+1}^-))+$$
$$+\sum_{e=1}^{p}\int_0^{\tau_{m+1}}\theta_e(Y(t))d\psi_e(t)+\sum_{i=1}^{k}K(Y(\tau_j^-),\xi_j)\Big]\quad(33)$$

Letting $m\to M$ *we have*

$$\phi(y)\geq E^{y}\Big[\int_0^{\tau_{\mathcal{S}}}f(Y^w(t),u(t)))dt+g(Y^w(\tau_{\mathcal{S}}))\chi_{\{\tau_{\mathcal{S}}<\infty\}}+$$
$$+\int_0^{\tau_{\mathcal{S}}}\theta(Y^w(t))d\psi_l(t)+\sum_{i=1}^{k}K(Y(\tau_j^-),\xi_j)\Big]=J^w(y)\quad\textit{for all } y\in\mathcal{S}\qquad(34)$$

If we assume that conditions $2(i)-(vi)$ *hold, and apply the above reasoning to*
$\hat{w}=(\hat{v},\hat{\xi},\hat{u})$, *then we get the following equalities, from* (33) *and* (34), *respectively,*

$$\phi(y)+\sum_{j=1}^{m}E^{y}\Big[\{\mathcal{M}\phi(Y(\tau_j^-))-\phi(Y(\tau_j^-))\}\chi_{\{\tau_j<\tau_{\mathcal{S}}\}}\Big]=E^{y}\Big[\phi(Y(\tau_{m+1}^-))$$
$$-\int_0^{\tau_{m+1}}\mathcal{L}\phi(Y(t)))dt-\int_0^{\tau_{m+1}}\sum_{i=1}^{k}\frac{\partial\phi}{\partial y_i}(Y(t^-)\sum_{l=1}^{p}\kappa_{ie}(Y(t^-))d\hat{\psi}_e^{(c)}(t)$$
$$-\sum_{0<t_n<\tau_{j+1}}\triangle_{\psi}\phi(Y(t_n)+\sum_{i=1}^{k}K(Y(\tau_j^-),\xi_j)\Big]$$
$$=E^{y}\Big[\int_0^{\tau_{m+1}}f(Y(t),\hat{u}(t)))dt+\phi(Y(\tau_{m+1}^-))+$$
$$+\sum_{e=1}^{p}\int_0^{\tau_{m+1}}\theta_e(Y(t))d\psi_e(t)+\sum_{i=1}^{k}K(Y(\tau_j^-),\xi_j)\Big]\quad(35)$$

$$\phi(y) = E^y\Big[\int_0^{\tau_\mathcal{S}} f(Y^{\hat{w}}(t), \hat{u}(t)))dt + g(Y^{\hat{w}}(\tau_\mathcal{S}))\chi_{\{\tau_\mathcal{S}<\infty\}} +$$
$$+ \int_0^{\tau_\mathcal{S}} \theta(Y^{\hat{w}}(t))d\hat{\psi}_e(t) + \sum_{i=1}^{k} K(Y^{\hat{w}}(\tau_j^-), \hat{\xi}_j)\Big] = J^{\hat{w}}(y) \quad \textit{for all } y \in \mathcal{S}. \quad (36)$$

Consequently, we obtain

$$\phi(y) = \Phi(y) = \sup\{J^{(w)}(y); w \in \mathcal{W}\} = J^{\hat{w}}(y). \quad (37)$$

This completes the proof of the theorem.

4 Application

We now illustrate the application of the verification theorem for the general combined singular and impulse control theory.

Example 4.1 { Optimal harvesting policy under transaction costs}

Suppose that if there are no interventions the stochastic process $X(t)$, which might represent the remaining resources, for example some mineral resource or wildlife population at time t (with $\mu, \sigma, \beta > 0$ constants), evolves according to

$$dX(t) = \mu dt + \sigma dB(t) + \beta \int_{\Re} z\tilde{N}(dt, dz); \;\; X(0) = x > 0 \quad (38)$$

where $B(t)$ is 1-dimensional standard Brownian motion, $\tilde{N}(.,.)$ is a compensated Poisson random measure and $\beta z \leq 0$ for $a.a\ z(\nu)$.

Now, assume that at any time τ_j, the investor is free to take out an amount, ξ_j, from $X(t)$ and such a transaction incurs a cost, denoted by $m(\xi_j)$, and given by

$$m(\xi_j) = \delta\xi_j + c$$

where $c \geq 0$ and $\delta \in (0, 1)$ are constants.

We suppose that the decision maker applies a combined control $w = (v, u, \psi)$ where

$$v := (\tau_1, \tau_2, ..., \tau_j..., \xi_1, \xi_2, ..., \xi_j...)$$

is an impulse control, $u(t)$ is the absolutely contiuous control and $\psi(t)$ is an increasing, adapted cádlág process representing the total amount taken out from $X(t)$ up to time t. ψ is a singular control.

Let $\mathcal{W}$ be the set of all combined singular and impulse controls $w = (v, u, \psi)$ such that $X^{(w)}(t) \geq 0$. We call $\mathcal{W}$ the set of admissible combined singular and impulse controls.

We now assume that as a result of applying the combined singular and impulse control w the evolution of the controlled process $X(t) = X^{(w)}(t)$ is described by (39) – (41) given below

$$X^{(w)}(t) = X(t) \qquad \text{if} \quad 0 \leq t < \tau_1; \tag{39}$$

$$dX^{(w)}(t) = \mu dt + \sigma dB(t) + \beta \int_{\Re} z\tilde{N}(dt, dz) \quad - \quad (1+\delta)d\psi(t)$$

$$\text{if} \quad \tau_j \leq t < \tau_{j+1}; \tag{40}$$

$$X^{(w)}(\tau_j) = \check{X}^{(w)}(\tau_j^-) - (1+\delta)\xi_j - c. \tag{41}$$

Define the performance criterion, $J^{(w)}(s, x)$, by

$$J^{(w)}(s, x) := E^{s,x}\Big[\int_0^{\tau} e^{-\rho(s+t)} X^{\alpha}(t) d\psi(t)\Big] \tag{42}$$

where $\tau = \inf\{t : X(t) \leq 0\}$ (time to exhaustion of resourcs), $\rho > 0$ is a discount factor, $0 <\mid \alpha \mid \leq 1$, $E(.)$ denotes expectation with respect to probability law P and $w = (v, u, \psi)$ represents an admissible combined singular and impulse control.

The problem is to find $\Phi(s, x)$ and $w^* = (v^*, u^*, \psi^*)$ such that

$$\Phi(s, x) = \sup_{(v,u,\psi)=w\in\mathcal{W}} J^{(w)}(s, x) = J^{(w^*)}(s, x) \tag{43}$$

Solution

This is an example of a combined singular and impulse control problem in a jump diffusion market. In this case the singular control is ψ since $d\psi(t)$ may be singular with respect to the Lebesgue measure dt. The impulse control is v.

We apply Theorem 3.1 to solve the problem. Here we separate cases .

Case 1: $0 < \alpha \leq 1$

This case is a jump diffusion extension of Example 3.2 in Leirvik (2005) and the analyses, particularly of the singular features of the problem, are similar in some stages . However, it is worthwhile to note that in Leirvik (2005) the diffusion version of the problem is formulated and solved using singular control theory only whereas in in this work we use the more general combined singular and impulse control for Lévy processes.

It can easily be observed that in light of Theorem 3.1 we have
$K = u = g = f = 0$, $\theta = e^{-\rho s}x^{\alpha}$, $\kappa(s,x) = -(1+\delta)$,
$\Gamma(s,x,\xi) = x - (1+\delta)\xi - c$ and $\mathcal{S} = \{(s,x);\ x > 0\}$.

If there are no interventions, the generator of the controlled process

$$Y(t) = \begin{bmatrix} s \\ X(t) \end{bmatrix}; \quad Y(0) = y = \begin{bmatrix} 0 \\ x \end{bmatrix}$$

coincides with the second order integro-partial differential operator, $\mathcal{L}$, given by

$$\mathcal{L}\phi(s,x) = \frac{\partial \phi}{\partial s} \quad + \quad \mu\frac{\partial \phi}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 \phi}{\partial x^2} + \int_{\Re}\{\phi(s,x+\beta z) - \phi(s,x) - \beta z\frac{\partial \phi}{\partial x}\}\nu(dz).$$

We suggest a solution of the form

$$\phi(s,x) = e^{-\rho s}\varphi(x).$$

With this solution candidate we have

$$\mathcal{L}\phi(s,x) = e^{-\rho s}\mathcal{L}'\varphi(x),$$

where

$$\mathcal{L}'\varphi(x) = -\rho\varphi(x) + \mu\varphi'(x) + \frac{1}{2}\sigma^2\varphi''(x) + \int_{\Re}\{\varphi(x+\beta z) - \varphi(x) - \beta z\varphi'(x)\}\nu(dz).$$

In this example the intervention operator, $\mathcal{M}$, is given by

$$\begin{aligned}\mathcal{M}\varphi(x) &= \sup_{\xi}\Big\{\varphi(x-(1+\delta)\xi-c);\ \xi\in\mathcal{Z}\Big\}\\ &= \sup_{\xi}\Big\{\varphi(x-(1+\delta)\xi-c);\ 0\le\xi\le\frac{x-c}{1+\delta}\Big\}\end{aligned}$$

and the continuation region is described as follows

$$\begin{aligned}D &= \{y\in\mathcal{S};\ \max\{\mathcal{M}\phi(y)-\phi(y),\max_e\{\sum_{i=1}^{k}\kappa_{ie}(y)\frac{\partial\phi}{\partial y_i}(y)+\theta_e(y)\}\}\le 0\}\\ &= \Big\{(s,x)\in\mathcal{S};\ \sup_{\xi}\Big\{\varphi(x-(1+\delta)\xi-c);\ 0\le\xi\le\frac{x-c}{1+\delta}\Big\}-\varphi(x)\le 0\Big\}\cap\\ \cap &\ \Big\{(s,x)\in\mathcal{S};\ -(1+\delta)\varphi'(x)+x^{\alpha}<0\Big\}.\end{aligned}$$

We propose that D be given by

$$D=\Big\{(s,x)\in\mathcal{S};\ 0<x<x^*\Big\}\text{ for some } x^*>0.$$

For all $x\in D$, condition $2(i)$ of Theorem 3.1 yields

$$-\rho\varphi(x)+\mu\varphi'(x)+\frac{1}{2}\sigma^2\varphi''(x)+\int_{\Re}\{\varphi(x+\beta z)-\varphi(x)-\beta z\varphi'(x)\}\nu(dz)=0.$$

To solve this last equation we try

$$\varphi(x)=e^{rx}\quad\text{for some } r\in\Re.$$

Then r must solve

$$h(r):=-\rho+\mu r+\frac{1}{2}\sigma^2r^2+\int_{\Re}\{e^{r\beta z}-1-r\beta z\}\nu(dz)=0. \tag{44}$$

Since $h(0)=-\rho<0$ and $\lim_{|r|\to\infty}h(r)=\infty$ we see that there exist two solutions $r_1,\ r_2$ of $h(r)=0$ such that $r_2<0<r_1$. Moreover, since $e^{r\beta z}-1-r\beta z\ge 0$ for all r,z we have $|\ r_2\ |>r_1$. With such a choice of r_1 and r_2 we try

$$\varphi(x)=A_1e^{r_1x}+A_2e^{r_2x}\quad\text{where } A_i\ (i=1,2) \text{ is a constant.}$$

We recall that $\phi(s,x) = e^{-\rho s}\varphi(x)$ is a value function and as such $\varphi(0) = 0$. This yields

$$A_1 = A = -A_2 > 0.$$

Thus

$$\varphi(x) = A(e^{r_1 x} - e^{r_2 x}); \quad 0 < x < x^*. \tag{45}$$

Outside D we require that

$$-(1+\delta)\varphi'(x) + x^\alpha = 0.$$

From this last equation we get

$$\varphi(x) = \frac{x^{1+\alpha}}{(1+\alpha)(1+\delta)} + A_3, \;\text{ for } x \geq x^* \tag{46}$$

where A_3 is an arbitrary constant. Combining (45) and (46) we get

$$\varphi(x) = \begin{cases} A(e^{r_1 x} - e^{r_2 x}); & 0 < x < x^* \\ \frac{x^{1+\alpha}}{(1+\alpha)(1+\delta)} + A_3, & \text{for } x \geq x^* \end{cases}.$$

To determine $A,\ A_3$ and x^* we need three equations. Using the fact that φ is continuous at x^* we obtain

$$A(e^{r_1 x^*} - e^{r_2 x^*}) = \frac{(x^*)^{1+\alpha}}{(1+\alpha)(1+\delta)} + A_3. \tag{47}$$

Also, since $\varphi \in C^1$ at $x = x^*$ then

$$A(r_1 e^{r_1 x^*} - r_2 e^{r_2 x^*}) = \frac{(x^*)^\alpha}{1+\delta}. \tag{48}$$

From $\varphi \in C^2$ at $x = x^*$ we infer that

$$A(r_1^2 e^{r_1 x^*} - r_2^2 e^{r_2 x^*}) = \frac{\alpha (x^*)^{\alpha-1}}{1+\delta}. \tag{49}$$

Using (48) and (49) we get

$$e^{(r_1 - r_2)x^*} = \frac{r_2(r_2 x^* - \alpha)}{r_1(r_1 x^* - \alpha)}. \tag{50}$$

Now, x^* is determined by solving the equation

$$x^* = \frac{\alpha r_1 - \alpha r_2 e^{(r_2-r_1)x^*}}{r_1^2 - r_2^2 e^{(r_2-r_1)x^*}}. \tag{51}$$

The value of x^* obtained from this last equation is used to find A and A_3 from equations (49) and (47) by substitution.

We now examine $\mathcal{M}\varphi$.

$$\begin{aligned}
\mathcal{M}\varphi(x) &= \sup_{\xi}\Big\{\varphi(x-(1+\delta)\xi - c,);\ \ 0 \le \xi \le \frac{x-c}{1+\delta}\Big\} \\
&= \sup_{\xi}\Big\{A(e^{r_1[x-(1+\delta)\xi-c]} - e^{r_2[x-(1+\delta)\xi-c]});\ \ 0 \le \xi \le \frac{x-c}{1+\delta}\Big\} \\
&\le \sup_{\xi}\Big\{A(e^{r_1 x} - e^{r_2 x}.e^{-r_2(1+\delta)\xi}.e^{-r_1 c});\ \ 0 \le \xi \le \frac{x-c}{1+\delta}\Big\} \\
&\le A(e^{r_1 x} - e^{r_2 x}). \\
&= \varphi(x).
\end{aligned}$$

This shows that the point of maximum , $\hat{\xi}(x)$, is given by

$$\hat{\xi}(x) = 0.$$

The above results are summarised in the following theorem

Theorem 4.1 *Let $X(t)$ be given by (39) – (41) and $J^{(w)}(s,x)$ be defined by (42). Assume that $\rho \ge 0$ and $0 < \alpha \le 1$. Then*

$$\Phi(s,x) := \sup_{w\in W} J^{(w)}(s,x) = \begin{cases} Ae^{-\rho s}(e^{r_1 x} - e^{r_2 x}); & 0 < x < x^* \\ e^{-\rho s}\Big(\frac{x^{1+\alpha}}{(1+\alpha)(1+\delta)} + A_3\Big), & for\ x \ge x^* \ . \end{cases}$$

where x^* A and A_3 are determined by solving equations (51), (49) and (47) simultaneously. In this case it is optimal not to exercise impulse control. The corresponding optimal singular control $(\hat{u}, \hat{\psi})$ is as follows

- *If $x \le x^*$ it is optimal to do nothing.*
- *If $x > x^*$ it is optimal to take out an amount $\xi = x - x^*$*

Proof 4.1 *In this proof we verify that the function $\phi(s,x)$ given by*

$$\phi(s,x) := \sup_{w\in W} J^{(w)}(s,x) = \begin{cases} Ae^{-\rho s}(e^{r_1 x} - e^{r_2 x}); & if \quad 0 < x < x^* \\ e^{-\rho s}\Big(\frac{x^{1+\alpha}}{(1+\alpha)(1+\delta)} + A_3\Big), & if \quad x \ge x^*. \end{cases}$$

where x^*, A *and* A_3 *are determined by* (51), (49) *and* (47) , *satisfies all the requirements of Theorem* 3.1.

It is not difficult to observe that $\phi(s,x)$ *is continuous on* $\bar{\mathcal{S}}$ *and also differentiable on* $\mathcal{S}$. *Thus, condition* $1(i)$ *is satisfied.*

We now show that $\phi \geq \mathcal{M}\phi$ *on* $\mathcal{S}$. *But for* $x \leq x^*$ *we have*

$$\begin{aligned}
\mathcal{M}\varphi(x) &= \sup_{\xi}\Big\{\varphi(x-(1+\delta)\xi - c,);\;\; 0 \leq \xi \leq \frac{x-c}{1+\delta}\Big\} \\
&= \sup_{\xi}\Big\{A(e^{r_1[x-(1+\delta)\xi-c]} - e^{r_2[x-(1+\delta)\xi-c]});\;\; 0 \leq \xi \leq \frac{x-c}{1+\delta}\Big\} \\
&\leq \sup_{\xi}\Big\{A(e^{r_1 x} - e^{r_2 x}.e^{-r_2(1+\delta)\xi}.e^{-r_2 c});\;\; 0 \leq \xi \leq \frac{x-c}{1+\delta}\Big\} \\
&\leq A(e^{r_1 x} - e^{r_2 x}). \\
&= \varphi(x).
\end{aligned}$$

For $x \geq x^*$ *we note that*

$$\begin{aligned}
\mathcal{M}\varphi(x) &= \sup_{\xi}\Big\{\varphi(x-(1+\delta)\xi - c,);\;\; 0 \leq \xi \leq \frac{x-c}{1+\delta}\Big\} \\
&= \sup_{\xi}\Big\{\frac{[x-(1+\delta)\xi-c]^{1+\alpha}}{(1+\alpha)(1+\delta)} + A_3\;\; 0 \leq \xi \leq \frac{x-c}{1+\delta}\Big\} \\
&\leq \frac{x^{1+\alpha}}{(1+\alpha)(1+\delta)} + A_3 \\
&= \varphi(x).
\end{aligned}$$

Hence, $\phi \geq \mathcal{M}\phi$. *Thus, condition* $1(ii)$ *is satisfied. To verify* $1(iii)$ *we just have to show that* $-(1+\delta)\varphi'(x) + x^{\alpha} \leq 0$ *since* $e^{-\rho s} > 0$ *for all* $s \geq 0$. *For* $x \geq x^*$ *this condition holds by construction of* φ.

Now, for $x \in D$ *we refer to Example* 3.2 *in* [8].

This proves that condition $1(iii)$ *is satisfied.*

The process $Y^w(t)$ *spends no time on the boundary of* D, *that is to say* $\chi_{\partial D}(Y^{(w)}(t)) = 0$ *a.e. Consequently,*

$$E^y\Big[\int_0^{\tau_{\mathcal{S}}} \chi_{\partial D}(Y^{(w)}(t))dt\Big] = 0 \quad \textit{for all } y \in \mathcal{S},\; w \in \mathcal{W}$$

Thus, $1(iv)$ *is satisfied.*

We note that $\partial D = x^*$. So $\partial D = x^*$ is a Lipstchitz surface since it is a constant and this verifies condition $1(v)$.

To prove condition $1(vi)$, let us consider the intervals

$$I_1 := (0, \delta),\ I_2 := (x^* - \delta, x^*) \text{ and } I_3 := (x^*, x^* + \delta)$$

where δ is an arbitrarily small positive number.

For $x \in I_1$ we have

$$0 \leq \varphi'(x) = A(r_1 e^{r_1 x} - r_2 e^{r_2 x}) < \ (r_1 e^{\delta r_1} - r_2),$$

and

$$-A(r_2^2 e^{\delta r_1} + r_2^2) \leq \varphi''(x) = \ A(r_1^2 e^{r_1 x} - r_2^2 e^{r_2 x}) < A r_1^2 e^{\delta r_1}.$$

If $x \in I_2$ we have

$$0 \leq \varphi'(x) = A(r_1 e^{r_1 x} - r_2 e^{r_2 x}) < \ A(r_1 e^{r_1 x^*} - r_2),$$

whereas

$$-A(r_2^2 e^{r_2 x^*} + r_2^2) \leq \varphi''(x) = A((r_1^2 e^{r_1 x} - r_2^2 e^{r_2 x}) < A r_1^2 e^{r_1 x^*}.$$

Finally, taking $x \in I_3$ then

$$0 \leq \varphi'(x) = A(r_1 e^{r_1 x} - r_2 e^{r_2 x}) < \ (r_1 e^{r_1 (x^* + \delta)} - r_2),$$

and

$$-A(r_2^2 e^{r_2 (x^* + \delta)} + r_2^2) \leq \varphi''(x) = \ (r_1^2 e^{r_1 x} - r_2^2 e^{r_2 x}) < A r_1^2 e^{r_1 (x^* + \delta)}.$$

Using these results we can conclude that φ has locally bounded first and second order derivatives near ∂D, and so condition $1(vi)$ is verified. The rest of the conditions of Theorem 3.1 hold by construction of φ.

Conclusion

Since $\phi(s,x) = e^{-\rho s}\varphi(x)$ satisfies all the conditions of Theorem 3.1 we conclude that

$$\phi(s,x) = \Phi(s,x) = J^{w^*}(s,x)$$

and the optimal strategy is to wait until the time, τ_j, that the resouces reach or exceed x^* and then take out an amount ξ_{τ_j} given by

$$\xi_{\tau_j} = \max\{x(\tau_j) - x^*, 0\}$$

Case 2: $-1 \leq \alpha < 0$

Without loss of generality we consider $\alpha = -\frac{1}{2}$. In this case the performance functional, $J^w(s,x)$, is given by

$$J^{(w)}(s,x) := E\Big[\int_s^\tau e^{-\rho(s+t)}(X(t))^{-\frac{1}{2}} d\psi(t)\Big].$$

Just like in the previous case we observe that $K = u = g = f = 0$, $\theta = e^{-\rho s}x^{-\frac{1}{2}}$, $\kappa(s,x) = -(1+\delta)$, $\Gamma(s,x,\xi_j) = x - (1+\delta)\xi_j - c$ and $\mathcal{S} = \{(s,x);\ x > 0\}$. It is worthwhile to note that $\theta : \Re \rightarrow \Re$ is a non-increasing function (density dependent prices) and for that reason our analysis follows closely arguments presented in [2], with the necessary extensions to the jump diffusion case, and also, in our case the discussion takes transaction costs into account. Additionally, here we examine the problem as a combined singular and impulse control whereas in [2] it is handled from the singular control angle only.

If we apply the *"take the money and run "*-strategy, $\dot{w}$, then *all* the resources are taken out immediately. Such a strategy is described by

$$\dot{w}(s) = \dot{\psi}(s) = (1-\delta)x - c. \tag{52}$$

The value function obtained from this strategy is

$$\Phi(s,x) = e^{-\rho s}x^{-\frac{1}{2}}[(1-\delta)x - c] = e^{-\rho s}[(1-\delta)\sqrt{x} - cx^{-\frac{1}{2}}]; \quad x > 0. \tag{53}$$

Apparently, this strategy is not optimal simply because it does not take into account the impact of transaction costs on total discounted gains, neither does it cater for the price increases as the resources diminish. Consequently, we seek a kind of "chattering strategy", denoted by $\tilde{w}^{(m,\eta)} = \tilde{\psi}^{(m,\eta)}$ where m is a fixed positive integer and $\eta > 0$.

At times τ_j given by

$$\tau_j = \Big(s + \frac{j}{m}\eta\Big) \wedge \tau : \quad j = 1, 2, ..., m \tag{54}$$

an amount of resources $\triangle\tilde{\psi}(\tau_j)$ given by

$$\triangle\tilde{\psi}(\tau_j) := \tilde{\xi}_j = \frac{1}{m}x$$

is taken out. This gives the expected total value of harvested resources

$$J^{(\tilde{w})}(s,x) = E^{s,x}\left[\sum_{j=1}^{m} e^{-\rho\tau_j}[(X^{(\tilde{w})}(\tau_j^-))^+]^{-\frac{1}{2}}\right]\tilde{\xi}_j \tag{55}$$

where

$$x^+ = \max\{x, 0\}; \quad x \in \Re. \tag{56}$$

We may present this as

$$J^{(\tilde{w})}(s,x) = E^{s,x}\left[\sum_{j=1}^{m} e^{-\rho\tau_j}[(x-(1+\delta)\xi_j - c)^+]^{-\frac{1}{2}}\right]\tilde{\xi}_j \tag{57}$$

Letting $\eta \to 0$ we realise that $\tau_j \to s$ for $j = 1, 2, ..., m$ and we get

$$\begin{aligned} J^{(\tilde{w}(m,0))}(s,x) : &= \lim_{\eta\to 0} J^{(\tilde{w}(m,\eta))}(s,x) \\ &= \lim_{\eta\to 0} E^{(s,x)}\left[\sum_{j=1}^{m} e^{-\rho\tau_j}[(x - \frac{j}{m}(1+\delta)x - c)^+]^{-\frac{1}{2}}\right]\frac{x}{m} \\ &= e^{-\rho s}\sum_{j=1}^{m} h(x_j)\triangle x_j. \end{aligned}$$

where $h(y) = [(x-(1+\delta)y-c)^+]^{-\frac{1}{2}},\ x_j = \frac{jx}{m}$ and $\triangle x_j = x_{j+1} - x_j = \frac{x}{m}$.

Given $\epsilon > 0$ there exists a positive integer m such that

$$e^{-\rho s} \mid \int_0^x [(x-(1+\delta)y-c)^+]^{-\frac{1}{2}}dy - \sum_{j=1}^{m} h(x_j)\triangle x_j \mid < \epsilon. \tag{58}$$

By making an appropriate choice of m and η we obtain the following

$$\mid J^{(\tilde{w}(m,\eta))}(s,x) - e^{-\rho s}\int_0^x [(x-(1+\delta)y-c)^+]^{-\frac{1}{2}}dy \mid < \epsilon. \tag{59}$$

We conclude that

$$\lim_{\substack{m\to\infty \\ \eta\longrightarrow 0}} J^{\tilde{w}}(s,x) = e^{-\rho s}\int_0^x [(x-(1+\delta)y-c)^+]^{-\frac{1}{2}}dy = \frac{2e^{-\rho s}}{1+\delta}\sqrt{x-c}. \tag{60}$$

We call this "chattering policy" of applying $\tilde{w}^{(m,\eta)}$ in the limit as $\eta \to 0$ and $m \to \infty$ the *policy of immediate chattering down to 0.*

Let us now investigate whether the function

$$\phi(s,x) := \frac{2e^{-\rho s}}{1+\delta}\sqrt{x-c}.$$

satifies the conditions of Theorem 3.1.

Condition $1(i)$ holds since the function

$$\phi(s,x) := \frac{2e^{-\rho s}}{1+\delta}\sqrt{x-c}$$

is differentiable on S and continuous on the closure of S whenever $x-c>0$. To investigate condition $1(ii)$ we observe that

$$\begin{aligned}\mathcal{M}\phi &= \sup_{\xi}\Big\{\phi(\Gamma(s,x,\xi)) : \;\; 0 \le \xi \le \frac{x-c}{1+\delta}\Big\} \\ &= \frac{2e^{-\rho s}}{1+\delta}\sup_{\xi}\Big\{\sqrt{x-(1+\delta)\xi - c} : \;\; 0 \le \xi \le \frac{x-c}{1+\delta}\Big\} \\ &\le \frac{2e^{-\rho s}}{1+\delta}\sqrt{x-c} \\ &= \phi(s,x,).\end{aligned}$$

Hence, $\phi(s,x,)$ satisfies condition $1(ii)$.

To find out whether $\phi(s,x,)$ satisfies condition $1(iii)$ we proceed as follows

$$\begin{aligned}\sum_{i=1}^{k}\kappa_{ie}(y)\frac{\partial\phi}{\partial y_i}(y) + \theta_e(y) &= -(1+\delta).\frac{2e^{-\rho s}}{1+\delta}.\frac{d}{dx}\Big[(x-c)^{\frac{1}{2}}\Big] + e^{-\rho s}x^{-\frac{1}{2}} \\ &= -e^{-\rho s}\Big[-(x-c)^{-\frac{1}{2}} + x^{-\frac{1}{2}}\Big] \\ &\le e^{-\rho s}\Big[-\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{x}}\Big] \\ &= 0.\end{aligned}$$

This proves that $\phi(s,x,)$ satisfies condition $1(iii)$.

Using the second-order integro-partial-differential operator

$$\begin{aligned}\mathcal{L}\phi(s,x) &= \frac{\partial\phi}{\partial s} + \mu\frac{\partial\phi}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2\phi}{\partial x^2} \\ &+ \int_{\Re}\{\phi(s,x+\beta z) - \phi(s,x) - \beta z\frac{\partial\phi}{\partial x}\}\nu(dz),\end{aligned}$$

we obtain

$$\begin{aligned}
\mathcal{L}\phi(s,x) &= \frac{e^{-\rho s}}{1+\delta}\Big[-2\rho(x-c)^{\frac{1}{2}} + \mu(x-c)^{-\frac{1}{2}} - \frac{1}{4}\sigma^2(x-c)^{-\frac{3}{2}} \\
&\quad + \int_{\Re}\{2\sqrt{x+\beta z-c} - 2(x-c)^{\frac{1}{2}} - \beta z(x-c)^{-\frac{1}{2}}\}\nu(dz)\Big] \\
&\le \frac{e^{-\rho s}}{1+\delta}\Big[-2\rho(x-c)^{\frac{1}{2}} + \mu(x-c)^{-\frac{1}{2}} - \frac{1}{4}\sigma^2(x-c)^{-\frac{3}{2}} \\
&\quad + \int_{\Re}\{2\sqrt{(x-c)} - 2(x-c)^{\frac{1}{2}} - \beta z(x-c)^{-\frac{1}{2}}\}\nu(dz)\Big] \\
&= \frac{e^{-\rho s}}{1+\delta}\Big[-2\rho(x-c)^{\frac{1}{2}} + \mu(x-c)^{-\frac{1}{2}} - \frac{1}{4}\sigma^2(x-c)^{-\frac{3}{2}} \\
&\qquad\qquad - \int_{\Re}\beta z(x-c)^{-\frac{1}{2}}\nu(dz)\Big].
\end{aligned}$$

We have applied the fact that $\beta z \le 0$. Thus

$$\begin{aligned}
\mathcal{L}\phi(s,x) &\le \frac{-2\rho e^{-\rho s}}{1+\delta}(x-c)^{-\frac{3}{2}}[(x-c)^2 - \frac{\mu}{2\rho}(x-c) + \frac{\sigma^2}{8\rho} + (x-c)\int_{\Re}\beta z\nu(dz)] \\
&= \frac{-2\rho e^{-\rho s}}{1+\delta}(x-c)^{-\frac{3}{2}}\Big[(x-c)^2 + \Big(\int_{\Re}\beta z\nu(dz) - \frac{\mu}{2\rho}\Big)(x-c) + \frac{\sigma^2}{8\rho}\Big].
\end{aligned}$$

So, condition $1(vii)$ holds if $x \ge c$ and

$$\Big(\int_{\Re}\beta z\nu(dz) - \frac{\mu}{2\rho}\Big)^2 \le \frac{\sigma^2}{2\rho}.$$

We now state the following result:

Theorem 4.2 *Let $X^{(w)}(t)$ be given by $(39)-(41)$.*

1. *Assume that $x \ge c$ and*

$$\left(\int_{\Re}\beta z\nu(dz) - \frac{\mu}{2\rho}\right)^2 \le \frac{\sigma^2}{2\rho}. \tag{61}$$

Then

$$\Phi(s,x) := \frac{2e^{-\rho s}}{1+\delta}\sqrt{x-c}. \tag{62}$$

where σ and ρ are defined as before. This value is achieved in the limit if we apply the strategy $\tilde{w}^{(m,\eta)}$ described above with $\eta \to 0$ and $m \to \infty$, that is, by applying the policy of immediate chattering to 0.

2. If

$$\left(\int_{\Re}\beta z\nu(dz)-\frac{\mu}{2\rho}\right)^2>\frac{\sigma^2}{2\rho}. \tag{63}$$

then the value function has the form

$$\Phi(s,x)=\begin{cases} e^{-\rho s}A(e^{r_1x}-e^{r_2x}); & for \quad 0\le x<x^* \\ e^{-\rho s}(\frac{2}{1+\delta}\sqrt{x-c}-\frac{2}{1+\delta}\sqrt{x^*-c}+B) & for \quad x^*\le x \end{cases} \tag{64}$$

for some constants $A>0,\ B>0$ and $x^*>0$ where r_1 and r_2 are the solutions of the equation

$$-\rho+\mu r+\frac{1}{2}\sigma^2r^2+\int_{\Re}\{e^{r\beta z}-1-r\beta z\}\nu(dz)=0. \tag{65}$$

with $r_2<0<r_1$ and $|\,r_2\,|>r_1$.

In both cases 1. and 2. the corresponding optimal policy is the following:

- *If $x>x^*$ it is optimal to apply immediate chattering from x down to x^*.*
- *If $0<x<x^*$ it is optimal to apply the harvesting equal to the local time of the downward reflected process $\bar{X}(t)$ at x^*.*

Proof 4.2 *We need to show that the proposed value function satisfies all the conditions of Theorem 3.1. Let us first examine the case*

$$\left(\int_{\Re}\beta z\nu(dz)-\frac{\mu}{2\rho}\right)^2\le\frac{\sigma^2}{2\rho}.$$

In this case we have

$$\phi(s,x):=\frac{2e^{-\rho s}}{1+\delta}\sqrt{x-c}.$$

From the construction of $\phi(s,x)$ we can state that conditions $1(i)-(iii)$ and $1(vii)$ are satisfied.

Since $X(t)$ spends no time on ∂D, then $\chi_{\partial D}X(t) = 0$ a.e and this leads to

$$E^y\Big[\int_0^\tau \chi_{\partial D}(Y^{(v)}(t))dt\Big] = 0 \quad \textit{for all } y \in \mathcal{S},\ v \in \mathcal{V}.$$

So, condition $1(iv)$ is satisfied.

In this example the boundary, ∂D, of the non-intervention region, D, is given by

$$\partial D = \partial D_1 \cup \partial D_2$$

where $\partial D_1 = \{0\}$ and $\partial D_2 = \{x^*\}$. But ∂D_1 and ∂D_2 are both Lipschitz surfaces since each of them is a singleton which consists of a constant. Hence, ∂D is also a Lipschitz surface. T hus $\phi(s,x)$ satisfies condition $1(v)$.

For $x > c$ it can easily be verified that the function

$$\phi(s,x) := \frac{2e^{-\rho s}}{1+\delta}\sqrt{x-c}.$$

is twice continuously differentiable on $\mathcal{S} \setminus \partial D$ and non of its derivatives explodes near ∂D. This establishes the requirements of $1(vi)$.

Recalling that $g\Big(Y^w(\tau)\Big) = 0$ we note that

$$\lim_{t\to\tau}\phi(t,x) = \lim_{t\to\tau}\frac{2e^{-\rho t}}{1+\delta}\sqrt{x-c} = 0 = g\Big(Y^w(\tau)\Big) \quad as \quad \tau \to \infty.$$

This proves that $1(vii)$ is satisfied.

The remaining conditions in part 1. of Theorem 3.1 can also be verified without much difficulty.

Up to this point we have proved that

$$\phi(s,x) \geq \Phi(s,x).$$

Now, by construction of $\phi(s,x)$ we observe that

$$\mathcal{L}^{\hat{\psi}(y)}\phi + f(y,\hat{\psi}(y)) = 0 \quad for \;\; all \;\; y \in D.$$

That is to say, $2(i)$ is satisfied.

Conditions $2(ii) - 2(vii)$ can be verified using similar arguments as in Example 2.14 of [7].

So, in this case

$$\phi(s,x) := \frac{2e^{-\rho s}}{1+\delta}\sqrt{x-c}.$$

satisfies all the requirements of Theorem 3.1. *Hence it is a value function for the given problem.*

We now treat the case

$$\left(\int_{\Re}\beta z\nu(dz) - \frac{\mu}{2\rho}\right)^2 > \frac{\sigma^2}{2\rho}.$$

For this case we need to show that the function $\phi(s,x)$ *given by*

$$\phi(s,x) = \begin{cases} e^{-\rho s}A(e^{r_1 x} - e^{r_2 x}); & \text{for} \quad 0 \le x < x^* \\ e^{-\rho s}\Big(\frac{2}{1+\delta}\sqrt{x-c} - \frac{2}{1+\delta}\sqrt{x^*-c} + B\Big) & \text{for} \quad x^* \le x \end{cases} \tag{66}$$

also satisfies conditions of Theorem 3.1 *where* $A,\ B,\ x^*,\ r_1,\ r_2$ *are as specified in Theorem* 4.2

Here we follow closely arguments used to prove part (b) of Theorem 3.2 in [2], *where we effect the necessary extension arguments to cater for the jump component as well as transaction costs. First, we observe that if we apply the policy of immediate chattering from* x *to* x^* *where* $0 < x^* < x$, *then the value of the dividends paid out is given by*

$$e^{-\rho s}\int_0^{x-x^*}[(x-(1+\delta)y-c)^+]^{-\frac{1}{2}}dy = \frac{2e^{-\rho s}}{1+\delta}\Big[\sqrt{x-c} - \sqrt{((1+\delta)x^* - \delta x - c)^+}\Big].$$

This follows by the argument $(53)-(58)$ *presented above. To verify the conclusions of part* 2 *of Theorem* 4.2 *we observe that* r_1 *and* r_2 *are the roots of the equation*

$$-\rho + \mu r + \frac{1}{2}\sigma^2 r^2 + \int_{\Re}\{e^{r\beta z} - 1 - r\beta z\}\nu(dz) = 0.$$

Hence, by defining $\phi(s,x)$ *as in* (66) *it is relatively easy to show that for* $x < x^*$

$$\mathcal{L}\phi(s,x) = 0 \tag{67}$$

and

$$\phi(s,0)=0. \tag{68}$$

Combining the smooth contact principle and the requirement that $\phi(s,x)$ *be* C^2 *at* $x=x^*$, *we obtain the following three equations*

$$A(e^{r_1x^*}-e^{r_2x^*})=B \tag{69}$$

$$A(r_1e^{r_1x^*}-r_2e^{r_2x^*})=(x^*)^{-\frac{1}{2}} \tag{70}$$

$$A(r_1^2e^{r_1x^*}-r_2^2e^{r_2x^*})=-\frac{1}{2}(x^*)^{-\frac{3}{2}} \tag{71}$$

Dividing (69) *by* (70) *we obtain the equation*

$$\frac{r_1e^{r_1x^*}-r_2e^{r_2x^*}}{r_1^2e^{r_1x^*}-r_2^2e^{r_2x^*}}=-2x^* \tag{72}$$

Now, observing that

$$\lim_{x^*\to 0}\frac{r_1e^{r_1x^*}-r_2e^{r_2x^*}}{r_1^2e^{r_1x^*}-r_2^2e^{r_2x^*}}=\frac{1}{r_1+r_2}<0 \tag{73}$$

and

$$\lim_{x^*\to \infty}\frac{r_1e^{r_1x^*}-r_2e^{r_2x^*}}{r_1^2e^{r_1x^*}-r_2^2e^{r_2x^*}}=\frac{1}{r_1}>0 \tag{74}$$

the intermediate value theorem guarantees the existence of x^* *satisfying equation* (72) *With this value of* x^* *we define* A *by* (70) *and* B *by* (69). *We have proved the existence of a solution of the system* (69) $-$ (71) *where* $A>0,\ B>0,\ x^*>0$. *With this choice of* $A>0,\ B>0,\ x^*>0$ *the function* $\phi(s,x)$ *becomes a* C^2 *and we can easily verify that* ϕ *satisfies conditions* $1(i)-(x)$ *of Theorem* 3.1. *Hence,*

$$\phi(s,x)\geq\Phi(s,x) \qquad \textit{for all } s,x. \tag{75}$$

Moreover, the non-intervention region D *given by* (3.1) *is identified to be*

$$D=\{(s,x):\ 0<x<x^*\}. \tag{76}$$

Consequently, by (66) *we know that condition* $2(i)$ *of Theorem* 3.1 *holds.*

Additionally, it is an established fact that the local time $\hat{\psi}$ *of the downward reflected process* $\bar{X}(t)$ *at* x^* *satisfies conditions* $2(ii)-2(vii)$ *(see, for example* [2] *and* [9] *and references therein).*

By Theorem 3.1 *we conclude that if* $x \leq x^*$ *then*

$$\psi^* := \hat{\psi}$$

is optimal and

$$\phi(s,x) = \Phi(s,x).$$

Finally, if $x > x^*$ *then it follows by* (4.28) *that immediate chattering from* x *to* x^* *gives the value*

$$\Phi(s,x) \geq e^{-\rho s}[\sqrt{x}(1-\delta) - cx^{-\frac{1}{2}}] + \Phi(s,x^*) \qquad \textit{for all } x > x^* \tag{77}$$

Combining this with (74) *this proves that*

$$\phi(s,x) = \Phi(s,x) \qquad \textit{for all } s, x \tag{78}$$

and the proof of part 2 *of Theorem* 4.2 *is complete.*

5 Conclusion

In this paper we have presented the combined singular and impulse control problem for jump diffusion processes. A verification theorem was formulated and proved for jump diffusions. An example on optimal dividend payout/optimal harvesting was analysed and explicit solutions were derived. In the example both proportional and fixed transaction costs were considered.

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