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An Extension of the Maximum Principle and its Application to a System of Parabolic Equations

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Abstract

In the article, the quasi-linear parabolic equation $u_t = u_{xx} + \epsilon u_{yy} + f(t,x,y,u)$ is approximated by $O(h^2)$ finite difference method with respect to the variable y. As the result, the system of n parabolic equations is obtained in terms of the variables t and x. An extension of the maximum principle is stated and proved for the system of parabolic equations. The principle is used to prove uniform convergence of the method. Applying the maximum principle, the global error estimate is given. The system of parabolic equations is solved by the implicit finite difference method combined with Gauss-Seidel iterative method. The *Mathematica* module *SolveParabolic* is designed and used for testing the methods. Examples of the quasi-linear diffusion equations have been solved and the numerical results are given in the tables 1,2,3. The results confirm effectiveness of the methods with the small global error on the level $O(h^2)$.

1 The Initial Boundary Problem

We shall consider the following semi-linear parabolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \epsilon \frac{\partial^2 u}{\partial y^2} + f(t,x,y,u), \qquad (t,x,y) \in Q, \qquad \epsilon > 0 \tag{1}$$

with the initial conditions

$$u(0,x,y) = g(x,y), \qquad (x,y) \in \Omega \tag{2}$$

and with boundary condition

$$u(t,x,y) = \phi(t,x,y), \qquad (t,x,y) \in \partial Q \tag{3}$$

where $\Omega = \{0 < x, y < 1\}$, is the unit square with the boundary $\partial\Omega$. We have $Q = \{(t, x, y) : (x, y) \in \Omega,\ t \geq 0\}$ and $\partial Q = \{(t, x, y) : (x, y) \in \Omega,$ when $t = 0,$ and $(x, y) \in \partial\Omega,$ when $t > 0\}$.

We assume that $f(t, x, y, u)$, $g(x, y)$ and $\phi(t, x, y)$ are continuous functions for $(x, y) \in \Omega \cup \partial\Omega$, $-\infty < u < \infty$ and $t \geq 0$.
In the case when $\epsilon = 0$, equation (1) represents the family of diffusion equations in one space variable x with the parameter y.
Let us note that Nirenberg's maximum principle holds for equation (1) and it yields the unique regular solution $u(t, x, y)$ under the initial boundary conditions (2) and (3).

1.1 Linearization

Let $f(t, x, y, u)$ be twice continuously differentiable function with respect to $u \in (-\infty, \infty)$. We shall write equation (1) in the neighbourhood of $u = 0$ in the following linearized forms:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \epsilon \frac{\partial^2 u}{\partial y^2} + f(t, x, y, 0) + \frac{\partial f(t, x, y, \eta_u)}{\partial u} u, \tag{4}$$

or

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \epsilon \frac{\partial^2 u}{\partial y^2} + f(t, x, y, 0) + \frac{\partial f(t, x, y, 0)}{\partial u} u + \frac{1}{2} \frac{\partial^2 f(t, x, y, \xi_u)}{\partial u^2} u^2. \tag{5}$$

for certain $\eta_u, \xi_u \in (-\infty, \infty)$.
In order to investigate stability and convergence of the method of planes, we shall apply the following version of Nirenberg's maximum principle.

1.2 Nirenberg's Maximum Principle

Every regular solution $v(t, x)$ of the parabolic equation

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + c(t, x)v + r(t, x), \qquad c(t, x) \leq 0,\ \ r(t, x) \leq 0,$$

or of the inequality

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} - c(t, x)v \geq 0, \qquad 0 \leq x \leq 1,\ \ t \geq 0,$$

attains its maximum values in Q, at the boundary, ∂Q, if it is non-negative (c.f. [1]).

2 The $O(h^2)$ Method

Let us consider heat equation (1) with the initial conditions (2) and boundary equations (3). Substituting into equation (1)

$$\epsilon \frac{\partial^2 u(t,x,y_i)}{\partial y^2} = \epsilon \Lambda u(t,x) - \frac{\epsilon h^2}{12}\frac{\partial^4 u(t,x,\xi_i)}{\partial y^4}, \qquad \xi_i \in (y_{i-1}, y_{i+1}),$$

we obtain the following semi discrete scheme

$$\begin{aligned} \frac{\partial u_i(t,x)}{\partial t} &= \frac{\partial^2 u_i(t,x)}{\partial x^2} + \epsilon \Lambda u_i(t,x) \\ &+ f_i(t,x,y_i,v_i) + \psi_i^y(t,x,h), \quad i = 1,2,...,N, \end{aligned} \tag{6}$$

with boundary conditions

$$u_i(t,x) = \phi(t,x,y_i) \quad 0 \le x \le 1, \quad i = 0, N+1, \quad t \ge 0, \tag{7}$$

and the initial conditions

$$u_i(0,x,y_i) = g(x,y_i), \; i = 0,1,\ldots,N+1. \tag{8}$$

Here $y_i = ih, \; u_i(t,x) = u(t,x,y_i), \; (N+1)h = 1$, and

$$\Lambda u_i(t,x) = \frac{u_{i-1}(t,x) - 2u_i(t,x) + u_{i+1}(t,x)}{h^2},$$

where the local truncation error

$$\psi_i^y(t,x,h) = -\frac{\epsilon h^2}{12}\frac{\partial^4 u(t,x,\xi_i)}{\partial y^4},$$

at a certain point $\xi_i \in (\xi_{i-1}, \xi_{i+1})$.

Re-writing the above scheme in the matrix form, we get:

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} + \epsilon M_0 U + F + \Psi, \tag{9}$$

with boundary conditions

$$U(t,x) = \Phi(t,x), \;\; \text{where}, \;\; \Phi = (\phi_1, \phi_2, \ldots, \phi_{\mathrm{N}}),$$

and initial conditions

$$U(0,x) = G(x), \;\; \text{and} \;\; G = (g_1, g_2, ,\ldots, g_N).$$

We have

$$U = (u_1, u_2, \ldots, u_N),$$

$$F = (f_1 + \frac{\epsilon\phi_0}{h^2}, f_2, f_3, \ldots, f_N + \frac{\epsilon\phi_{N+1}}{h^2}),$$

$$f_i = f(t, x, y_i, v_i), \ i = 1, 2, \ldots, N,$$

$$\Psi = (\psi_1, \psi_2, \ldots, \psi_N).$$

Here Ψ is the vector form of the truncation error. If we neglect the truncation error in equation (9), we get.

$$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2} + \epsilon M_0 V + F, \tag{10}$$

with the same boundary and initial conditions as (9). Here $V = (v_1, v_2, \ldots, v_N)$. The matrix

$$-M_0 = \frac{1}{h^2}\begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 \end{bmatrix}_{N\times N}$$

Let us note that for $\epsilon = 0$, the system of parabolic differential equations (10) is the same as equation (1) considered for the coordinates $y_j = jh, \quad j = 1, 2, \cdots, N$. So that we have

$$\frac{\partial u(t, x, y_j)}{\partial t} = \frac{\partial^2 u(t, x, y_j)}{\partial x^2} + f(t, x, y_j, u_j), \tag{11}$$

for $j = 1, 2, \cdots, N$.

Now, we shall state and prove the extension of Nirenberg's maximum principle to the system of parabolic equations (10).

2.1 Extension of the Maximum Principle

Theorem 1 *Assume that $f(t, x, y, u)$ has the derivative $\frac{\partial f}{\partial u} \leq 0$. If $V(t, x)$ is the regular solution of the system of parabolic equations (10), then the length*

$$R(t, x) = \sqrt{v_1^2(t, x) + v_2^2(t, x) + \ldots + v_N^2(t, x)}$$

of the solution $V(t,x)$ attains its maximum value in Q_T at the boundary ∂Q_T, provided that $(V,F_0) \le 0$, where

$$(V,F_0) = (v_1 f_{10} + v_2 f_{20} + \ldots + v_N f_{n0}),$$

$$f_{i0} = f(t,x,y_i,0),\ i = 1,2,...,N,$$

$$Q_T = \{(t,x) : 0 \le x \le 1,\ 0 \le t \le T\},$$

$$\partial Q_T = \{(t,x) : 0 \le x \le 1, \quad \textit{when}\, t = 0,\ \textit{or}\, x = 0,1,\ 0 \le t \le T\}.$$

Proof: Let $e(t,x) = [e_1(t,x), e_2(t,x), \ldots, e_N(t,x)]$ be the unit vector in the direction of the solution $V(t,x)$, when $V(t,x) \ne 0$. We note that the function $R(t,x)$ satisfies the equation

$$\frac{\partial R}{\partial t} = \frac{\partial^2 R}{\partial x^2} + \Phi(t,x)R + (e,F), \tag{12}$$

where the quadratic form

$$\Phi(t,x) = \epsilon(M_0 e, e) - \left(\frac{\partial e}{\partial x}, \frac{\partial e}{\partial x}\right),$$

is non-positive definite, that is $\Phi(t,x) \le 0$ for all t,x. Indeed, let us substitute $V = eR$ to (10). Then, we compute

$$\begin{aligned}
\frac{\partial V}{\partial t} &= e\frac{\partial R}{\partial t} + R\frac{\partial e}{\partial t}, \\
\frac{\partial V}{\partial x} &= e\frac{\partial R}{\partial x} + R\frac{\partial e}{\partial x}, \\
\frac{\partial^2 V}{\partial x^2} &= e\frac{\partial^2 R}{\partial x^2} + 2\frac{\partial e}{\partial x}\frac{\partial R}{\partial x} + R\frac{\partial^2 e}{\partial x^2}.
\end{aligned}$$

Hence, by substitution to (10), we find

$$e\frac{\partial R}{\partial t} + R\frac{\partial e}{\partial x} = e\frac{\partial^2 R}{\partial x^2} + 2\frac{\partial e}{\partial x}\frac{\partial R}{\partial x} + R\frac{\partial^2 e}{\partial x^2} + R\epsilon M_0 e + F.$$

Multiplying the above by e, we have

$$\begin{aligned}
(e,e)\frac{\partial R}{\partial t} + R\left(e, \frac{\partial e}{\partial t}\right) &= (e,e)\frac{\partial^2 R}{\partial x^2} + 2\frac{\partial R}{\partial x}\left(\frac{\partial e}{\partial x}, e\right) \\
&+ \left[\left(e, \frac{\partial^2 e}{\partial x^2}\right) + \epsilon\,(M_0 e, e)\right] R + (e,F).
\end{aligned} \tag{13}$$

Because

$$(e,e)=1, \qquad (e,\frac{\partial e}{\partial t})=0,$$
$$(e,\frac{\partial e}{\partial x})=0, \quad (e,\frac{\partial^2 e}{\partial x^2})=-(\frac{\partial e}{\partial x},\frac{\partial e}{\partial x}),$$

the equation (13) takes the scalar form

$$\frac{\partial R}{\partial t}=\frac{\partial^2 R}{\partial x^2}+\left[\epsilon(M_0e,e)-\left(\frac{\partial e}{\partial x},\frac{\partial e}{\partial x}\right)\right]R+(e,F). \tag{14}$$

The matrix $-M_0$ is positive definite. So that $(M_0e,e)\le 0$, and the coefficient at R

$$\epsilon(M_0e,e)-\left(\frac{\partial e}{\partial x},\frac{\partial e}{\partial x}\right)\le 0,$$

since $\left(\frac{\partial e}{\partial x},\frac{\partial e}{\partial x}\right)\ge 0$. Now, we note that

$$(e,F)=e_1f_1+e_2f_2+\ldots+e_nf_N,$$
$$f_i=f(t,x,y_i,v_i)=f(t,x,y_i,0)+\frac{\partial f(t,x,y_i,\xi_i)}{\partial u}v_i,$$

for a certain $\xi_i, \;\; i=1,2,\ldots,N$.
Hence, by the assumption, we obtain the inequality

$$\begin{aligned}(e,F) &= e_1f_{10}+e_2f_{20}+\ldots+e_nf_{N0}\\ &+ e_1\frac{\partial f_1}{\partial u}v_1+e_2\frac{\partial f_2}{\partial u}v_2+\ldots+e_N\frac{\partial f_N}{\partial u}v_N\\ &= \frac{1}{R}(V,F_0)+\frac{1}{R}(\frac{\partial f_1}{\partial u}v_1^2+\frac{\partial f_2}{\partial u}v_2^2+\ldots+\frac{\partial f_N}{\partial u}v_N^2)\le 0.\end{aligned}$$

By Nirengberg's maximum principle, the length $R(t,x)$ attains its maximum value in Q_T at the boundary ∂Q_T.

Now, we shall state and prove the theorem on estimate of the solution $V(t,x)$ of the system of parabolic equations (10) by data in the initial boundary conditions and by the right hand side vector function $F(t,x)$.

Theorem 2 *If $V(t,x)$ is the regular solution of the system of parabolic equations (10), then the length $R_V\le\sqrt{v_1^2+v_2^2+\ldots+v_N^2}$ of the solution $V(t,x)$ satisfies the following inequality*

$$R_V(t,x)\le 2\max_Q(F,F)^{\frac{1}{2}}+\max_{\partial Q}R_V(t,x), \qquad (t,x)\in Q\cup\partial Q.$$

Proof. Let us note that the length $R_V(t,x)$ of the vector $V(t,x)$ satisfies equation (12). We shall prove the theorem in the case when $f(t,x,y,u) = c(t,x)u + r(t,x)$ is a linear function with respect to u, using the following lemma.

Lemma 1 *Let $v(t,x)$ be a regular solution of the equation*

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + c(t,x)v + r(t,x), \qquad c(t,x) \le 0. \tag{15}$$

If there exists a function $k(t,x)$ such that

1. $-(\frac{\partial^2 k}{\partial x^2} - \frac{\partial k}{\partial t}) \ge \max_Q |r(t,x)|, \quad (t,x) \in Q,$

2. $k(t,x) \ge \max_{\partial Q} |v(t,x)|, \qquad (t,x) \in \partial Q,$

then

$$|v(t,x)| \le k(t,x) \quad \textit{for all} \quad (t,x) \in Q \cup \partial Q.$$

Proof of lemma 1. Let us consider the auxiliary functions

$$q_1(t,x) = v(t,x) - k(t,x),$$
$$q_2(t,x) = -v(t,x) - k(t,x).$$

By assumption 2,

$$\begin{aligned} q_1(t,x) &= v(t,x) - k(t,x) \le 0, \quad (t,x) \in \partial Q,\\ q_2(t,x) &= -v(t,x) - k(t,x) \le 0, \quad (t,x) \in \partial Q. \end{aligned} \tag{16}$$

By assumption 1, for $c(t,x) \le 0$ and $k(t,x) \ge 0$, we have

$$\begin{aligned} \frac{\partial^2 q_1}{\partial x^2} - \frac{\partial q_1}{\partial t} + c(t,x)q_1 &= \frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial t} + c(t,x)v \\ &- [\frac{\partial^2 k}{\partial x^2} - \frac{\partial k}{\partial t} + c(t,x)k] \\ &= -r(t,x) - [\frac{\partial^2 k}{\partial x^2} - \frac{\partial k}{\partial t}] - c(t,x)k \\ &\ge -r(t,x) + \max_Q |r(t,x)| \ge 0. \end{aligned}$$

Similarly

$$\begin{aligned}\frac{\partial^2 q_2}{\partial x^2} - \frac{\partial q_2}{\partial t} + c(t,x)q_2 &= -[\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial t} + c(t,x)v] - [\frac{\partial^2 k}{\partial x^2} - \frac{\partial k}{\partial t} + c(t,x)k] \\ &= -r(t,x) - [\frac{\partial^2 k}{\partial x^2} - \frac{\partial k}{\partial t}] - c(t,x)k) \\ &\geq -r(t,x) + \max_Q |r(t,x)| \geq 0.\end{aligned}$$

Hence, by the maximum principle, the inequalities (16) hold for all $(t,x) \in Q$. Therefore

$$-k(t,x) \leq v(t,x) \leq k(t,x), \qquad (t,x) \in Q \cup \partial Q,$$

or

$$|v(t,x)| \leq k(t,x), \qquad (t,x) \in Q \cup \partial Q.$$

This ends the proof of lemma 1.

Let us set in (15), $v(t,x) = R_V(t,x)$, $c(t,x) = \Phi(t,x)$ and $r(t,x) = (e,F)$. Then, the function $R_V(t,x)$ satisfies the equations (8) and (15). In order to complete the proof, we apply lemma 1 for the function

$$k(t,x) = (3 - Exp(x)) \max_Q |(e,F)| + \max_{\partial Q} R_V(t,x).$$

We note that the function $k(t,x)$ satisfies the conditions 1 and 2 in lemma 1. Indeed,

To 1

$$-\left(\frac{\partial^2 k}{\partial x^2} - \frac{\partial k}{\partial t}\right) = Exp(x) \max_Q |(e,F)| \geq \max_Q |(e,F)|, \quad 0 \leq x \leq 1.$$

To 2

$$k(t,x) \geq \max_{\partial Q} R_V(t,x), \qquad (t,x) \in \partial Q.$$

By the thesis of lemma 1,

$$R_V(t,x) \leq k(t,x)$$

for all $(t,x) \in Q \cup \partial Q$.

Because

$$k(t,x) \leq 2 \max |(e,F)| + \max_{\partial Q} R_V(t,x) \leq 2 \max_Q (F,F)^{\frac{1}{2}} + \max_{\partial Q} R_V(t,x),$$

therefore

$$R_V(t,x) \leq 2 \max_Q (F,F)^{\frac{1}{2}} + \max_{\partial Q} R_V(t,x).$$

This end the proof of the theorem.