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ON FUZZY PRIME IDEALS OF LATTICE

*B.B.N. KOGUEP, **C. NKUIMI, ***C. LELE

Abstract. In this paper, We study the notion of fuzzy prime ideal and highlight the difference between fuzzy prime ideal and prime fuzzy ideal. Using the characterization of fuzzy ideal induced by a fuzzy set, we show that a fuzzy ideal μ of a lattice is a fuzzy prime ideal if and only if any nonempty $\alpha - cut$ set of μ is a prime ideal. We show that if a fuzzy ideal μ is proper and maximal then μ is a fuzzy prime ideal. By an example, we show that the converse is not true. At the end we give a fuzzy prime ideal theorem.

Key Words : Lattices, Ideals, filters, prime ideals, fuzzy sublattice, fuzzy ideal, fuzzy filters.

AMS classifications : 03G10, 46H10, 06D50, 08A72

1. Backgrounds

Throughout this paper, L is a nonempty set and $\underline{L} = (L, \wedge, \vee, 0, 1)$ stands for a bounded distributive lattice.

Definition 1.1. [2] *A nonempty subset J of L is called an **ideal** of $\underline{L}$ if for all $x, y \in L$*

(i) *$x, y \in J$ imply $x \vee y \in J$.*

(ii) *if $x \in J$ with $y \leq x$, then $y \in J$.*

Definition 1.2. [2] *A nonempty subset F of L is called a **filter** of $\underline{L}$ if for all $x, y \in L$*

(i) *$x, y \in F$ imply $x \wedge y \in F$.*

(ii) *if $x \in F$ with $x \leq y$, then $y \in F$.*

Remark 1.3. *For any ideal J of $\underline{L}$, $0 \in J$. For any filter F of $\underline{L}$, $1 \in F$.*

Definition 1.4. [2] *Let A be a nonempty subset of L.*

(i) *The smallest ideal of $\underline{L}$ containing A is called the ideal of $\underline{L}$ generated by A, and is denoted by $A\downarrow$ or $\mathcal{I}_A$.*

(i) *the smallest filter of $\underline{L}$ containing A is called the filter of $\underline{L}$ generated by A, and is denoted by $A\uparrow$ or $\mathcal{F}_A$.*

Particulary if $A = \{x\}$, we denote $\{x\}\downarrow$ by $\mathcal{I}_x$ and $\{x\}\uparrow$ by $\mathcal{F}_x$.

Remark 1.5. [2] *Let A and B be subsets of L and a, b be elements of L.*

(i) $\mathcal{I}_0 = \{0\}$ *and* $\mathcal{I}_\emptyset = \{0\}$.

(ii) $\mathcal{I}_A = \{x \in L/\exists a_1, a_2, ..., a_n \in A, x \leq a_1 \vee a_2 \vee ... \vee a_n\}$.

(iii) $A \subseteq B \Rightarrow \mathcal{I}_A \subseteq \mathcal{I}_B$.

(iv) $a \leq b \Rightarrow \mathcal{I}_a \subseteq \mathcal{I}_b$

(v) *If A is an ideal of $\underline{L}$, then $\mathcal{I}_A = A$.*

Definition 1.6. [2] *Let J and F be respectively a proper ideal and a proper filter of $\underline{L}$.*

(i) *J is said to be* ***prime*** *if $a, b \in L$ and $a \wedge b \in J$ imply $a \in J$ or $b \in J$.*

(ii) *F is said to be* ***prime*** *if $a, b \in L$ and $a \vee b \in F$ implies $a \in F$ or $b \in F$.*

Let $\mathcal{I}(\underline{L})$ and $\mathcal{F}(\underline{L})$ be respectively the set of all ideals and the set of all filters of $\underline{L}$. Then $(\mathcal{I}(\underline{L}), \subseteq, \cap, \cup)$ and $(\mathcal{F}(\underline{L}), \subseteq, \cap, \cup)$ are bounded distributive lattices.

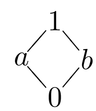
2. Fuzzy prime ideal of lattice

A **fuzzy subset** of L is a function $\mu : L \to [0, 1]$. Let μ be a fuzzy subset of L. For $\alpha \in [0, 1]$, the set $\mu_\alpha = \{x \in L/\mu(x) \geq \alpha\}$ is called a α **-level subset** of μ or α-**cut set** of μ, [7],[8].

Definition 2.1. $[1, definition\ 2.1]$ *A fuzzy subset μ of L is* ***proper*** *if it is a non constant function. A fuzzy subset μ such that $\mu(x) = 0$ for all $x \in L$ is an* ***improper*** *fuzzy subset.*

Definition 2.2. $[1, definition\ 2.2]$ *A fuzzy subset μ of L is called a* ***fuzzy sublattice*** *of $\underline{L}$ if $\mu(x \wedge y) \wedge \mu(x \vee y) \geq \mu(x) \wedge \mu(y)$ for all $x, y \in L$.*

Example 2.3. *Let L be the following boolean algebra,*



Consider $\mu = \{(0,1), (a,0), (b, \frac{1}{4}), (1,0)\}$ and $\nu = \{(0,1), (a,1), (b,1), (1,0)\}$. μ is a fuzzy sublattice of $\underline{L}$, but ν is not a fuzzy sublattice of $\underline{L}$ because $\nu(a \wedge b) \wedge \nu(a \vee b) = \nu(0) \wedge \nu(1) = 0$ and $\nu(a) \wedge \nu(b) = 1$, i.e., $\nu(a \wedge b) \wedge \nu(a \vee b) < \nu(a) \wedge \nu(b) = 1$.

Definition 2.4. $[1, definition\ 2.3; 5]$ *Let μ be a fuzzy sublattice of $\underline{L}$. Then*

(i) *μ is* ***a fuzzy ideal*** *of $\underline{L}$, if $\mu(x \vee y) = \mu(x) \wedge \mu(y)$ for all $x, y \in L$.*
(ii) *μ is a* ***fuzzy filter*** *of $\underline{L}$, if $\mu(x \wedge y) = \mu(x) \wedge \mu(y)$ for all $x, y \in L$.*

Example 2.5. *Consider the boolean algebra L as in example 2.3, $\mu = \{(0,1), (a, \frac{1}{5}), (b, \frac{1}{3}), (1, \frac{1}{5})\}$ and $\eta = \{(0,0), (a,1), (b,0), (1,1)\}$. μ is a fuzzy ideal of $\underline{L}$ and η is a fuzzy filter of $\underline{L}$.*

Proposition 2.6. [1] *Let μ be a fuzzy sublattice of $\underline{L}$. Then :*

(i) *μ is a fuzzy ideal of $\underline{L}$, if and only if, $x \leq y \Rightarrow \mu(x) \geq \mu(y)$, for all $x, y \in L$.*
(ii) *μ is a fuzzy filter of $\underline{L}$, if and only if, $x \leq y \Rightarrow \mu(x) \leq \mu(y)$, for all $x, y \in L$.*

Remark 2.7.
(i) *If μ is a fuzzy ideal of $\underline{L}$, then $\mu(0) \geq \mu(x) \geq \mu(1)$, for all $x \in L$.*
(ii) *If μ is a fuzzy filter of $\underline{L}$, then $\mu(0) \leq \mu(x) \leq \mu(1)$, for all $x \in L$.*

Theorem 2.8. $[1, lemma\ 3.1]$ *Let μ be a fuzzy subset of L. Then :*

(i) *μ is a fuzzy ideal of $\underline{L}$, if and only if, for any $\alpha \in [0;1]$, such that $\mu_\alpha \neq \emptyset$, μ_α is an ideal of $\underline{L}$.*
(ii) *μ is a fuzzy filter of $\underline{L}$, if and only if, for any $\alpha \in [0;1]$, such that $\mu_\alpha \neq \emptyset$, μ_α is a filter of $\underline{L}$.*

Corollary 2.9. *A nonempty subset I of L is an ideal of $\underline{L}$ if and only if the characteristic function of I is a fuzzy ideal of $\underline{L}$.*

Let $FI(\underline{L})$ and $FF(\underline{L})$ be respectively the set of all fuzzy ideals of $\underline{L}$ and the set of all fuzzy filters of $\underline{L}$.

Remark 2.10. *$(FI(\underline{L}), \leq)$ and $(FF(\underline{L}), \leq)$ are completely bounded distributive lattices, with $\leq$ defined by : for any $\mu, \nu \in FI(\underline{L})$ (or in $FF(\underline{L})$), $\mu \leq \nu \Leftrightarrow \mu(x) \leq \nu(x)$, for all $x \in \underline{L}$ and for any family $\{\mu_\alpha / \alpha \in \Lambda\}$ of fuzzy ideals (or fuzzy filters) of $\underline{L}$, $(\bigvee_{\alpha\in\Lambda} \mu_\alpha)(x) = \sup_{\alpha\in\Lambda} \mu_\alpha(x)$ and $(\bigwedge_{\alpha\in\Lambda} \mu_\alpha)(x) = \inf_{\alpha\in\Lambda} \mu_\alpha(x)$.*

We highlight the difference between the notion of prime fuzzy ideal and the notion of fuzzy prime ideal of a bounded distributive lattice.

Definition 2.11. [4] *(prime fuzzy ideal and prime fuzzy filter)*

(i) *A proper fuzzy ideal μ of $\underline{L}$ is called* ***prime fuzzy ideal*** *if for any two fuzzy ideals η, ν of $\underline{L}$, $\eta \wedge \nu \leq \mu \Rightarrow \eta \leq \mu$ or $\nu \leq \mu$.*

(ii) *A proper fuzzy filter μ of $\underline{L}$ is called* ***prime fuzzy filter*** *if for any two fuzzy filters η, ν of $\underline{L}$, $\eta \vee \nu \leq \mu \Rightarrow \eta \leq \mu$ or $\nu \leq \mu$.*

Definition 2.12. $[1, definition\ 4.1]$ *(fuzzy prime ideal and fuzzy prime filter)*

(i) *A proper fuzzy ideal μ of $\underline{L}$ is called* ***fuzzy prime ideal****, if $\mu(x \wedge y) \leq \mu(x) \vee \mu(y)$, for all $x, y \in L$.*

(ii) *A proper fuzzy filter μ of $\underline{L}$ is called* ***fuzzy prime filter****, if $\mu(x \vee y) \leq \mu(x) \vee \mu(y)$, for all $x, y \in L$.*

Example 2.13. *Consider the boolean algebra L as in example 2.3, $\mu = \{(0, \frac{1}{4}), (a, \frac{1}{6}), (b, \frac{1}{4}), (1, \frac{1}{6})\}$, $\eta = \{(0,0), (a,1), (b,0), (1,1)\}$ and $\nu = \{(0,1), (a,0), (b, \frac{1}{4}), (1,0)\}$. μ is a fuzzy prime ideal of $\underline{L}$, η is a fuzzy prime filter of $\underline{L}$ and ν is a fuzzy ideal, but not a fuzzy prime ideal of $\underline{L}$, since we have $\nu(a \wedge b) = \nu(0) = 1$ and $\nu(a) \vee \nu(b) = \frac{1}{4}$, i.e., $\nu(a \wedge b) > \nu(a) \vee \nu(b)$.*

With the next example, we show the difference between prime fuzzy ideal and fuzzy prime ideal.

Example 2.14. *Consider* $\mu = \{(0, \frac{3}{4}), (a, \frac{1}{3}), (b, \frac{3}{4}), (1, \frac{1}{3})\}$, $\eta = \{(0, 1), (a, \frac{1}{4}), (b, \frac{4}{5}), (1, \frac{1}{4})\}$ *and* $\nu = \{(0, \frac{3}{4}), (a, \frac{3}{4}), (b, \frac{2}{3}), (1, \frac{2}{3})\}$

η and ν are fuzzy ideals of $L = \{0, a, b, 1\}$ the boolean algebra L as in example 2.3 and μ is a fuzzy prime ideal of $\underline{L}$.

We have $(\eta \wedge \nu)(0) = \eta(0) \wedge \nu(0) = \nu(0) = \mu(0)$

$(\eta \wedge \nu)(a) = \eta(a) \wedge \nu(a) = \eta(a) = \frac{1}{4} < \mu(a) = \frac{1}{3}$

$(\eta \wedge \nu)(b) = \eta(b) \wedge \nu(b) = \nu(b) = \frac{2}{3} < \mu(b) = \frac{3}{4}$

$(\eta \wedge \nu)(1) = \eta(1) \wedge \nu(1) = \eta(1) = \frac{1}{4} < \mu(a) = \frac{1}{3}$.

Then $(\eta \wedge \nu) \leq \mu$. But $\eta(0) > \mu(0)$ and $\nu(a) > \mu(a)$. So μ is not a prime fuzzy ideal of $\underline{L}$.

Theorem 2.15. *Let $\mu \in FI(\underline{L})$. Then, μ is a fuzzy prime ideal of $\underline{L}$ if and only if for any $\alpha \in [0; 1]$, such that μ_α is a proper ideal of $\underline{L}$, μ_α is a prime ideal of $\underline{L}$.*

Proof: Suppose that μ is a fuzzy prime ideal of $\underline{L}$. Let $\alpha \in [0; 1]$, such that μ_α is a proper ideal of $\underline{L}$. Let $a, b \in L$, such that $a \wedge b \in \mu_\alpha$. Then $\mu(a \wedge b) \geq \alpha$. Therefore $\mu(a \wedge b) \geq \alpha \Rightarrow \mu(a) \vee \mu(b) \geq \alpha$ (since μ is a fuzzy prime ideal). Thus $\mu(a) \geq \alpha$ *or* $\mu(b) \geq \alpha$ (since any $\alpha \in [0, 1]$ is $\vee$ - irreducible). So $a \in \mu_\alpha$ *or* $b \in \mu_\alpha$. Using the fact that μ_α is an ideal of $\underline{L}$ (theorem 2.8), we conclude that μ_α is a prime ideal of $\underline{L}$.

Conversely suppose that for any $\alpha \in [0; 1]$ such that μ_α is a proper ideal of $\underline{L}$, μ_α is a prime ideal of $\underline{L}$. Let $x, y \in L$ and $\alpha = \mu(x \wedge y)$. We have $x \wedge y \in \mu_\alpha$. Hence $x \in \mu_\alpha$ or $y \in \mu_\alpha$ (since μ_α is a prime ideal). It follows that $\mu(x) \geq \alpha = \mu(x \wedge y)$ or $\mu(y) \geq \alpha = \mu(x \wedge y)$, then $\mu(x \wedge y) \leq \mu(x) \vee \mu(y)$. Therefore, μ is a fuzzy prime ideal of $\underline{L}$. □

Corollary 2.16. *A proper subset I of L is a prime ideal of $\underline{L}$, if and only if the characteristic function of I is a fuzzy prime ideal of $\underline{L}$.*

Proof: This is true since for any $\alpha \in]0;1]$, $(\chi_I)_\alpha = I$ and $(\chi_I)_0 = L$. □

Remark 2.17. *Theorem 2.15 and example 2.14 prove that there is no characterization of prime fuzzy ideal by its $\alpha-cut$ sets.*

Lemma 2.18. *Let μ be a fuzzy subset of L.*
Then $\mu(x) = \sup\{\alpha \in [0;1]/x \in \mu_\alpha\}$ for all $x \in L$.

Proof: Let $x \in L$. Let $\beta = \sup\{\alpha \in [0;1]/x \in \mu_\alpha\}$. For any $\epsilon > 0$, there is $\alpha_0 \in [0;1]$, such that $\beta - \epsilon < \alpha_0$ and $x \in \mu_{\alpha_0}$. Thus for any $\epsilon > 0$, $\beta - \epsilon < \mu(x)$. i.e., $\beta \leq \mu(x)$. Since $\mu(x) \in \{\alpha \in [0;1]/x \in \mu_\alpha\}$, we have $\mu(x) \leq \beta$. Then, $\mu(x) = \beta$. □

Let Γ be a subset of $[0,1]$.

Theorem 2.19. *Let $\{I_\alpha/\alpha \in \Gamma\}$ be a collection of ideals of $\underline{L}$ such that :*

(i) $L = \bigcup\limits_{\alpha\in\Gamma} I_\alpha$.
(ii) $\alpha > \beta$ *if and only if* $I_\alpha \subseteq I_\beta$ *for all* $\alpha, \beta \in \Gamma$.

Define a fuzzy subset ν of L by $\nu(x) = \sup\{\alpha \in \Gamma/x \in I_\alpha\}$ for all $x \in L$. Then ν is a fuzzy ideal of $\underline{L}$.

Proof: It is sufficient to prove that ν_α is an ideal of $\underline{L}$, for every $\alpha \in [0;1]$ with $\nu_\alpha \neq \emptyset$.

Let $\alpha \in [0;1]$. We have two cases : (1) $\alpha = \sup\{\beta \in \Gamma/\beta < \alpha\}$, (2) $\alpha \neq \sup\{\beta \in \Gamma/\beta < \alpha\}$.

Case (1) implies that $x \in \nu_\alpha \Leftrightarrow x \in I_\beta, for\ all\ \beta < \alpha,\ \beta \in \Gamma$. i.e., $x \in \bigcap\limits_{\substack{\beta\ <\alpha \\ \beta\in\Gamma}} I_\beta$. Hence $\nu_\alpha = \bigcap\limits_{\substack{\beta\ <\alpha \\ \beta\in\Gamma}} I_\beta$, which is an ideal of $\underline{L}$.

For the case (2), there exists $\epsilon > 0$ such that $[\alpha-\epsilon, \alpha] \cap \Gamma = \emptyset$. If $x \in \bigcup\limits_{\substack{\beta\ <\alpha \\ \beta\in\Gamma}} I_\beta$, then $x \in I_\beta$ for some $\beta \geq \alpha$. It follows that $\nu(x) \geq \beta \geq \alpha$ so that $x \in \nu_\alpha$. That is $\bigcup\limits_{\substack{\beta\ <\alpha \\ \beta\in\Gamma}} I_\beta \subseteq \nu_\alpha$. Conversely, if $x \notin \bigcup\limits_{\substack{\beta\ <\alpha \\ \beta\in\Gamma}} I_\beta$, then $x \notin I_\beta$ for all $\beta \geq \alpha$. Which implies that $x \notin I_\beta$ for all $\beta > \alpha - \epsilon$, that is, if $x \in I_\beta$ then $\beta \leq \alpha - \epsilon$. Thus $\nu(x) \leq \alpha - \epsilon$ and so $x \notin \nu_\alpha$. Consequently $\nu_\alpha = \bigcup_{\substack{\beta\ <\alpha \\ \beta\in\Gamma}} I_\beta$, which is an ideal of $\underline{L}$. □

Remark 2.20. *If* $\Gamma = \{0, 1\}$, *then* ν *is the characteristic function of* I_1.

Definition 2.21. *Let* μ *be a fuzzy subset of* L. *The* ***least fuzzy ideal*** *of* $\underline{L}$ *containing* μ *is called a* ***fuzzy ideal*** *of* $\underline{L}$ ***induced by*** μ *and denoted by* $< \mu >$.

Theorem 2.22. *Let* μ *be a fuzzy subset of* L. *Then the fuzzy subset* μ^* *of* L *defined by* $\mu^*(x) = \sup\{\alpha \in [0; 1]/x \in < \mu_\alpha >\}$ *for all* $x \in L$ *is the fuzzy ideal induced by* μ.

Proof: We first prove that for any $\lambda \in Im(\mu^*)$, μ^*_λ is an ideal of $\underline{L}$. For any $\lambda \in Im(\mu^*)$, let $\lambda_n = \lambda - \frac{1}{n}$ for any $n \in \mathbb{N}^*$, and let $x \in \mu^*_\lambda$. Then $\mu^*(x) \geq \lambda$, which implies that $\mu^*(x) > \lambda_n$, for any $n \in \mathbb{N}^*$. Hence there exist $\beta \in \{\alpha \in [0, 1]/x \in < \mu_\alpha >\}$ such that $\beta > \lambda_n$. Thus $\mu_\beta \subseteq \mu_{\lambda_n}$ and so $x \in < \mu_\beta > \subseteq < \mu_{\lambda_n} >$ for all $n \in \mathbb{N}^*$. Therefore, $x \in \bigcap_{n\in\mathbb{N}^*} < \mu_{\lambda_n} >$ Conversely, if $x \in \bigcap_{n\in\mathbb{N}^*} < \mu_{\lambda_n} >$, $\lambda_n \in \{\alpha \in [0, 1]/x \in < \mu_\alpha >\}$, for any $n \in \mathbb{N}^*$. Therefore, $\lambda_n = \lambda - \frac{1}{n} \leq \sup\{\alpha \in [0, 1]/x \in < \mu_\alpha >\} = \mu^*(x)$, for all $n \in \mathbb{N}^*$. Hence $\mu^*(x) \geq \lambda$, so that $x \in \mu^*_\lambda$. Then we have $\mu^*_\lambda = \bigcap_{n\in\mathbb{N}^*} < \mu_{\lambda_n} >$ which is an ideal of $\underline{L}$.

For any $x \in L$, let $\beta \in \{\alpha \in [0, 1]/x \in \mu_\alpha\}$. Then $x \in \mu_\beta$, and so $x \in < \mu_\beta >$. Thus $\beta \in \{\alpha \in [0, 1]/x \in < \mu_\alpha >\}$. Which implies that $\{\alpha \in [0, 1]/x \in \mu_\alpha\} \subseteq \{\alpha \in [0, 1]/x \in < \mu_\alpha >\}$. Then $\mu(x) \leq \mu^*(x)$. Therefore, $\mu \leq \mu^*$.

Finally let ν be a fuzzy ideal of $\underline{L}$ containing μ. Let $x \in L$, if $\mu^*(x) = 0$, then $\mu^*(x) \leq \nu(x)$. Assume that $\mu^*(x) = \lambda \neq 0$. Then $x \in \mu^*_\lambda = \bigcap_{n\in\mathbb{N}^*} < \mu_{\lambda_n} >$. i.e.,$x \in \mu_{\lambda_n}$, for all $n \in \mathbb{N}^*$; it follows that $\nu(x) \geq \mu(x) \geq \lambda_n = \lambda - \frac{1}{n}$, for all $n \in \mathbb{N}^*$; Then $\nu(x) \geq \lambda = \mu^*(x)$. Thus $\mu^* \leq \nu$. Hence, μ^* is the least fuzzy ideal of $\underline{L}$ containing μ. □

Example 2.23. *Let $\underline{L}$ the boolean algebra of example 2.3. Let μ be the fuzzy subset of L defined by $\mu = \{(0, \frac{1}{4}), (a, \frac{1}{5}), (b, \frac{1}{3}), (1, \frac{1}{6})\}$. Then, for any $\alpha \in [0; 1]$:*

$\alpha \in [0; \frac{1}{6}] \Rightarrow \mu_\alpha = L =< \mu_\alpha >$.

$\alpha \in]\frac{1}{6}; \frac{1}{5}] \Rightarrow \mu_\alpha = L - \{1\}$, *and* $< \mu_\alpha >= L$.

$\alpha \in]\frac{1}{5}; \frac{1}{4}] \Rightarrow \mu_\alpha = \{0, b\} =< \mu_\alpha >$.

$\alpha \in]\frac{1}{4}; \frac{1}{3}] \Rightarrow \mu_\alpha = \{b\}$, *and* $< \mu_\alpha >= \{0, b\}$.

$\alpha \in]\frac{1}{3}; 1] \Rightarrow \mu_\alpha = \emptyset$, *and* $< \mu_\alpha >= \{0\}$.

Therefore, $\mu^*(0) = \sup\{\alpha \in [0; 1]/0 \in< \mu_\alpha >\} = 1$, $\mu^*(a) = \sup\{\alpha \in [0; 1]/a \in< \mu_\alpha >\} = \frac{1}{5}$, $\mu^*(b) = \sup\{\alpha \in [0; 1]/b \in< \mu_\alpha >\} = \frac{1}{3}$ *and* $\mu^*(1) = \sup\{\alpha \in [0; 1]/x \in< \mu_\alpha >\} = \frac{1}{5}$.

Proposition 2.24. *For any $\mu \in FI(\underline{L})$, if μ is maximal in $FI(\underline{L})$ then for any $\alpha \in [0; 1]$ such that, μ_α is a proper ideal of $\underline{L}$, μ_α is maximal in $\mathcal{I}(\underline{L})$.*

Proof: Let $\mu \in FI(\underline{L})$ such that μ is maximal in $FI(\underline{L})$. Let $\alpha \in [0; 1]$ such that, μ_α is a proper ideal of $\underline{L}$. Let $J \in \mathcal{I}(\underline{L})$ such that $\mu_\alpha \subsetneqq J$. Suppose that $J \neq L$, then there is $a \in L$ such that $a \notin J$. Therefore, $a \notin \mu_\alpha$, i.e., $\mu(a) < \alpha$. Let γ be the fuzzy subset of L defined by $\gamma(x) = \mu(x)$ if $x \neq a$ and $\gamma(a) = \alpha$. Thus $\mu \leq \gamma \leq < \gamma >$ (the fuzzy ideal of $\underline{L}$ induced by γ). Since $\gamma \neq 1$, this is a contradiction to the fact that μ is maximal. Thus $J = L$. Therefore μ_α is maximal. □

Corollary 2.25. *Let $\mu \in FI(\underline{L})$. If μ is proper and maximal in $FI(\underline{L})$ then μ is a fuzzy prime ideal of $\underline{L}$.*

Proof: Suppose μ is proper and maximal in $FI(\underline{L})$. Then by proposition 2.24, μ_α is maximal for any $\alpha \in [0; 1]$ such that, μ_α is a proper ideal of $\underline{L}$. Therefore since in any bounded distributive lattice, maximal implies prime, we have for any $\alpha \in [0; 1]$, if μ_α is a proper ideal of $\underline{L}$, then μ_α is prime. Thus according to theorem 2.15, μ is a fuzzy prime ideal of $\underline{L}$. □

With the following example, we prove that the converse of proposition 2.24 is does not hold.

Let $L = \{0, a, b, 1\}$ be the boolean algebra of example 2.3. Then, $\mathcal{I}(\underline{L}) = \{\{0\}, \{0, a\}, \{0, b\}, L\}$.

Let μ be the fuzzy subset of L defined by $\mu(0) = 1$, $\mu(a) = \frac{1}{5}$, $\mu(b) = \frac{1}{3}$ and $\mu(1) = \frac{1}{5}$. μ is a fuzzy ideal of $\underline{L}$ and for any $\alpha \in [0; 1]$, we have :

$\alpha \in [0; \frac{1}{5}] \Rightarrow \mu_\alpha = L$, $\alpha \in]\frac{1}{5}; \frac{1}{3}] \Rightarrow \mu_\alpha = \{0, b\}$, $\alpha \in]\frac{1}{3}; 1] \Rightarrow \mu_\alpha = \{0\}$.

Since $\{0, b\}$ is a maximal ideal of $\underline{L}$, then for any $\alpha \in [0; 1]$ such that μ_α is a proper ideal of $\underline{L}$, μ_α is maximal in $\mathcal{I}(\underline{L})$.

Consider ν the fuzzy subset of L defined by $\nu(0) = 1$, $\nu(a) = \frac{1}{4}$, $\nu(b) = \frac{1}{3}$ and $\nu(1) = \frac{1}{4}$. Then, ν is a proper fuzzy ideal of $\underline{L}$, $\nu \neq \mu$ and $\mu \leq \nu$. Therefore, μ is not maximal.

We discuss the notion of prime fuzzy ideal theorem and fuzzy prime ideal theorem of bounded distributive complete lattice.

Theorem 2.26. [4, *Theorem* 1.4]*(**prime Fuzzy ideal theorem**) Let $\alpha \in [0; 1]$, μ be a fuzzy ideal of $\underline{L}$ and ν be a fuzzy filter of $\underline{L}$ such that $\mu \wedge \nu \leq \alpha$. Then there exists a fuzzy prime ideal γ of $\underline{L}$ such that $\mu \leq \gamma$ and $\gamma \wedge \nu \leq \alpha$.*

Theorem 2.27. *(**Fuzzy prime ideal theorem**) Let $\alpha \in [0; 1]$, μ be a fuzzy ideal of $\underline{L}$ and ν be a fuzzy filter of $\underline{L}$ such that $\mu \wedge \nu \leq \alpha$. Then there exists a fuzzy prime ideal γ of $\underline{L}$ such that $\mu \leq \gamma$ and $\gamma \wedge \nu \leq \alpha$.*

Proof: Put $I = \{x \in L / \mu(x) \nleq \alpha\}$ and $J = \{x \in L / \nu(x) \nleq \alpha\}$. Then I is an ideal of $\underline{L}$ and J is a filter of $\underline{L}$ ($I = \mu_\alpha$ and $J = \nu_\alpha$) such that $I \cap J = \emptyset$. Therefore, by Stone's prime ideal theorem there exists a prime ideal P of $\underline{L}$ such that $I \subseteq P$ and $P \cap J = \emptyset$. Consider γ a fuzzy subset of L defined by

$$\gamma(x) = \begin{cases} 1 & , if\ x \in P \\ \alpha & , if\ x \notin P \end{cases}$$

for all $x \in L$. Then we have to prove that γ is a fuzzy ideal of $\underline{L}$ and it is fuzzy prime. We have $\alpha \neq 1$, then γ is a proper fuzzy subset of L. Let

$x, y \in L$. If $x \le y$, then we have $x \notin P$ implies $y \notin P$ (because P is an ideal of $\underline{L}$); so $x \notin P$ implies $\gamma(x) \ge \gamma(y)$. On the other hand, $x \in P$ implies $\gamma(x) = 1 \ge \gamma(y)$. Therefore for any $x, y \in L$, $x \le y$ implies $\gamma(x) \ge \gamma(y)$. Let $x, y \in L$. Due to the result above, we have $\gamma(x \wedge y) \ge \gamma(x \vee y)$. We have $x \vee y \in P$ implies $x \in P$ and $y \in P$. Thus $\gamma(x \vee y) = \gamma(x) \wedge \gamma(y) = 1$. If $x \vee y \notin P$, then $x \notin P$ or $y \notin P$, hence $\gamma(x \vee y) = \gamma(x) \wedge \gamma(y) = \alpha$. Thus for any $x, y \in L$, $\gamma(x \wedge y) \ge \gamma(x \vee y) \ge \gamma(x) \wedge \gamma(y)$. Therefore, using proposition 2.6 we conclude that γ is a fuzzy ideal of $\underline{L}$. Finally, let $x, y \in L$. If $x \wedge y \in P$, then $x \in P$ or $y \in P$ (P is a prime ideal of $\underline{L}$), hence $\gamma(x) = 1$ or $\gamma(y) = 1$, i.e., $\gamma(x) \vee \gamma(y) = 1$. So $\gamma(x \wedge y) = \gamma(x) \vee \gamma(y)$. If $x \wedge y \notin P$, then $\gamma(x \wedge y) = \alpha \le \gamma(x) \vee \gamma(y)$. Therefore γ is a fuzzy prime ideal of $\underline{L}$. Let $x \in L$. If $x \in P$, then $\gamma(x) = 1$ and $\mu(x) \le \gamma(x)$. If $x \notin P$, then $x \notin I$ (since $I \subseteq P$), so $\mu(x) \le \alpha = \gamma(x)$. Therefore, $\mu \le \gamma$. Let $x \in L$. If $x \in P$, then $x \notin J$ (due to $P \cap J = \emptyset$), so $\nu(x) \le \alpha$ and $(\nu \wedge \gamma)(x) \le \alpha$. If $x \notin P$, then $\gamma(x) = \alpha$, so $(\nu \wedge \gamma)(x) \le \gamma(x) = \alpha$. Therefore, $\nu \wedge \gamma \le \alpha$. □

3. Conclusion and further suggestions

We have studied the notion of fuzzy prime ideals and established the fuzzy prime ideal theorem of lattice. One should observe that the concept of prime fuzzy ideal is different from the one of fuzzy prime ideals. In [5],[6] the prime ideal theorem in meet-hyperlatice and distributive hyperlaticce have been studied. One of the most promising ideas could be the investigation of fuzzy setting applied to hyperlattice.

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*Department of Mathematics, Faculty of Science, University of Yaounde 1, BP 812, Cameroon

E-mail address: koguep@yahoo.com

**Department of Mathematics,Faculty of Science, University of Yaounde 1, BP 812, Cameroon

E-mail address: nkuimi@yahoo.co.uk

*** Department of Mathematics, University of Dschang, BP 67, Cameroon

E-mail address: lele_clele@yahoo.com